Ward Identities and Radiative Rare Leptonic B-decays

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Standard Model process

$$B \rightarrow l\nu$$

- *Direct measurement of* f_B
- *CKM matrix element –* V_{ub}
- *New Physics beyond S.M.*

(at tree level)

The decay width

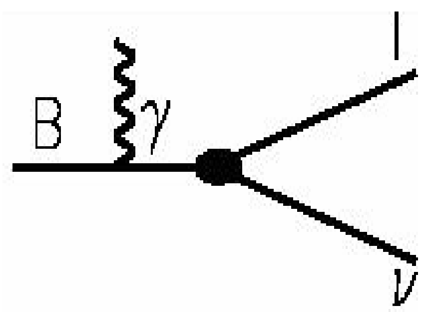
$$\Gamma\left(B \to l\nu\right) = \frac{G_F^2}{8\pi}\left|Vub\right|^2 f_B^2 \frac{m_l^2}{M_B^2} M_B^3 \left(1 - \frac{m_l^2}{M_B^2}\right)^2$$

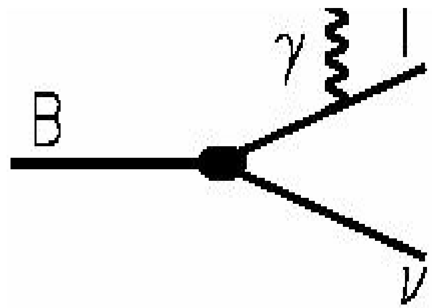
$$Br\left(B \to l\nu\right) \approx \begin{cases} 5.8\times10^{-12} & \text{for } e^- \\ 2.2\times10^{-7} & \text{for } \mu^- \end{cases}$$

The Radiative Partner $B \to \gamma l \nu$

In Radiative B-decay Process, there are two major contributions to the amplitude:

- ***Inner Bremsstrahlung (IB)***



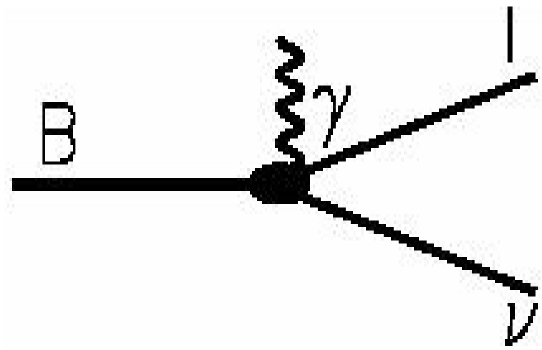


$$M_{IB} = ie\frac{G_F}{\sqrt{2}} V_{ub} f_B m_l \epsilon^*_\mu L^\mu$$

with

$$L^\mu = m_l \bar{u}(p_\nu)(1+\gamma_5)\left(\frac{2p^\mu}{2p\cdot k} - \frac{2p_l^\mu + \not{k}\gamma^\mu}{2p_l\cdot k}\right) v(p_l, s_l)$$

- ***Structure Dependent (SD)***



$$M_{SD} = -i\frac{G_F}{\sqrt{2}} V_{ub} f_B m_l \epsilon^*_\mu \tilde{H}^{\mu\nu} l_\nu$$

where

$$\tilde{H}^{\mu\nu} = iF_V(q^2)\epsilon^{\mu\nu\alpha\beta}k_\alpha p_\beta - F_A(q^2)(p \cdot k g^{\mu\nu} - p^\mu k^\nu)$$

$$l^\mu = \bar{u}(p_\nu)\gamma^\mu(1+\gamma_5)v(p_l, s_l)$$

$$q^\mu = (p-k)^\mu = (p_l + p_\nu)^\mu$$

It depends on vector and axial vector form factors.

The decay constant and form factors are defined as

$$\langle 0|\bar{u}\gamma^{\mu}\gamma_{5}b|B(p)\rangle = -if_{B}p^{\mu}$$

$$\langle\gamma(k)|\bar{u}\gamma^{\mu}\gamma_{5}b|B(p)\rangle = -[(\epsilon^{*}\cdot p)k^{\mu} - \epsilon^{*\mu}(p\cdot k)]F_{A}(q^{2})$$

$$\langle\gamma(k)|\bar{u}\gamma^{\mu}b|B(p)\rangle = -i\epsilon^{\mu\nu\alpha\beta}\epsilon^{*}_{\nu}p_{\alpha}k_{\beta}F_{V}(q^{2})$$

The Structure Dependent part is given by

$$iH^{\mu\nu} = i\int d^4x\, e^{ik\cdot x}\langle 0|T(j^{\mu}_{em}(x)J^{\nu}_{2}(0))|B(p)\rangle$$

For real photon we can write

$$H^{\mu\nu} = \tilde{H}^{\mu\nu} + f_B \frac{p^{\mu}p^{\nu}}{p\cdot k}$$

with $k_{\mu}\tilde{H}^{\mu\nu} = 0$

The absorptive part is given by

$$Abs[iH^{\mu\nu}] = \frac{1}{2}\int d^4x\, e^{ik\cdot x}\langle 0|[j^{\mu}_{em}(x), J^{\nu}_{2}(0)]|B(p)\rangle$$

$$= \frac{1}{2}(2\pi)^4 \Big[\sum_n \langle 0|j^{\mu}_{em}(0)|n\rangle\langle n|J^{\nu}_{2}(0)|B(p)\rangle\delta^4(k-p_n)$$

$$- \sum_n \langle 0|J^{\nu}_{2}(0)|n\rangle\langle n|j^{\mu}_{em}(0)|B(p)\rangle\delta^4(k+p_n-p)\Big]$$

The contribution to absorptive part are all possible intermediate states that couple to $B\gamma$ and are annihilated by the weak vertex $\langle 0|J_2^\nu(0)|n\rangle$ These include the multiparticle contrinum as well resonances with quantum numbers 1^- and 1^+.

$$F_V(t) = \frac{g_{BB^*\gamma}}{M_{B^*}^2 - t} f_{B^*} + \cdots$$

$$F_A(t) = \frac{f_{B_A^* B\gamma}}{M_{B_A^*}^2 - t} f_{B_A^*} + \cdots$$

$$\langle B^{*-}(q,\eta)\gamma(k,\epsilon) \mid B^-(P)\rangle = i g_{B^*B\gamma} \varepsilon_{\alpha\rho\mu\sigma} \epsilon^{*\alpha} q^\rho \eta^{*\mu} p^\sigma$$

$$\langle 0|i\bar{u}\sigma_{\mu\nu} b|B^{*-}(q,\eta)\rangle = f_T^{B^*}(q_\mu \eta_\nu - q_\nu \eta_\mu)$$

$$\langle B_A^{*-}(q,\eta)\gamma(k,\epsilon) \mid B^-(P)\rangle = i g_{B_A^* B\gamma}(\epsilon^*.\eta^*) - i f_{B_A^* B\gamma}(q.\epsilon^*)(k.\eta^*)$$

$$\langle 0|i\bar{u}\sigma_{\mu\nu} b|B_A^{*-}(q,\eta)\rangle = f_T^{B_A^*} \varepsilon_{\mu\nu\alpha\beta} \eta^\alpha q^\beta$$

We assume that the contributions from the radial excitations of B^* and B_A^* dominate the higher state contribution.

$$F_V(t) = \frac{R_V}{1-t/M_{B^*}^2} + \sum_i \frac{R_{V_i}}{1-t/M_{B_i^*}^2} + \frac{1}{\pi}\int_{S_0}^{M^2} \frac{\mathrm{Im}\,F_V^{\mathrm{Cont}}(s)}{s-t-i\varepsilon}\,ds$$

$$F_A(t) = \frac{R_A}{1-t/M_{B_A^*}^2} + \sum_i \frac{R_{A_i}}{1-t/M_{B_{A_i}^*}^2} + \frac{1}{\pi}\int_{S_0}^{M^2} \frac{\mathrm{Im}\,F_A^{\mathrm{Cont}}(s)}{s-t-i\varepsilon}\,ds$$

$$S_0 = M_B + m_\pi$$

where

$$R_V = \frac{g_{BB^*\gamma}}{M_{B^*}^2} f_{B^*}$$

$$R_A = \frac{f_{B_A^* B\gamma}}{M_{B_A^*}^2} f_{B_A^*}$$

If we model the continum contributions by quark triangle graph, we have

$$F_V^{\text{Cont}} = F_A^{\text{Cont}} = \frac{f_B}{M_B}\left\{\frac{Q_u}{\bar{\Lambda}} - \frac{Q_b}{M_B}\left(1 + \frac{\bar{\Lambda}}{M_B}\right)\right\}\frac{1}{1 - q^2/M_B^2}$$

where $\bar{\Lambda} = M_B - m_b$, together with the term

$$(Q_u - Q_b) f_B \frac{p^\mu p^\nu}{k \cdot p} = f_B \frac{p^\mu p^\nu}{k \cdot p}$$

To get constraints on the residues R_i it is useful to study the asymptotic behavior of form factors F_V and F_A .

$$iH^{\mu\nu} = i\int d^4x\, e^{iq\cdot x}\langle 0|T(j^{\mu}_{em}(0)J^{\nu}_{2}(x))|B(p)\rangle$$

$$iH^{\mu\nu} \underset{\vec{q}\to 0}{\overset{q_0\to i\infty}{\to}} i^2\left[\frac{1}{q_0}\int d^3x\langle 0|[J^{\nu}_{2}(0,\vec{x}),j^{\mu}_{em}(0)]|B(p)\rangle + \frac{i}{q_0^2}\int d^3x\left\langle 0\left|\left[\frac{\partial}{\partial t}J^{\nu}_{2}(t,\vec{x}),j^{\mu}_{em}(0)\right]_{t=0}\right|B(p)\right\rangle + \ldots\right]$$

$$j^{\mu}_{em} = \frac{2}{3}\bar{u}\gamma^{\mu}u - \frac{1}{3}\bar{b}\gamma^{\mu}b$$

$$J^{\nu}_{2} = \bar{u}\gamma^{\nu}(1-\gamma_5)b$$

In spirit of the current algebra, the equal time commutator in the first term can be evaluated to be

$$(Q_u + Q_b)\frac{1}{2}u^{\dagger}[O, O']b + (Q_u - Q_b)\frac{1}{2}u^{\dagger}[O, O']_+ b$$

where

$$O = \gamma^0 \gamma^\mu, \; O' = \gamma^0 \gamma^\nu (1 - \gamma^5)$$

Thus we can easily obtain

$$F_V(q^2) \to \frac{1}{3}\frac{f_B}{Q^2} = -\frac{1}{3}\frac{f_B}{q^2}$$

$$F_A(q^2) \to \frac{f_B}{q^2}$$

In heavy quark limit using the asymptotic behavior of equation

$$F\left(q^2\right) \to -\frac{1}{q^2}\left[RM^2 + \sum_i R_i M_i^2 + \frac{1}{\pi}\int_{S_0}^{M^2} \mathrm{Im} F_V^{\mathrm{Cont}}(s)\, ds\right]$$

the convergence condition restricts

$$RM^2 + \sum_i R_i M_i^2 + c = 0$$

$$c = \frac{1}{\pi}\int_{S_0}^{M^2} \mathrm{Im} F_V^{\mathrm{Cont}}(s)\, ds$$

Restriction to the first two radial excitations we have

$$RM^2 + R_1 M_1^2 + R_2 M_2^2 + c = 0$$

So we have from equation

$$F_V(t) = \frac{R_V}{1-t/M_{B^*}^2} + \sum_i \frac{R_{V_i}}{1-t/M_{B_i^*}^2} + \frac{1}{\pi}\int_{S_0}^{M^2} \frac{\mathrm{Im} F_V^{\mathrm{Cont}}(s)}{s-t-i\varepsilon}\, ds$$

$$F(q^2) = \frac{RM^2(M_2^2-M^2)}{(M^2-q^2)(M_2^2-q^2)} + \frac{R_1 M_1^2(M_2^2-M_1^2)}{(M_1^2-q^2)(M_2^2-q^2)}$$

$$+\frac{1}{M_2^2-q^2}\frac{1}{\pi}\int_{S_0}^{M^2}\frac{M_2^2-s}{s-q^2}\,\mathrm{Im}F_V^{\mathrm{Cont}}$$

$$F(q^2) = \frac{RM^2(M_2^2-M^2)}{(M^2-q^2)(M_2^2-q^2)} + \frac{R_1 M_1^2(M_2^2-M_1^2)}{(M_1^2-q^2)(M_2^2-q^2)}$$

$$+\frac{M_2^2-M^2}{(M_2^2-q^2)(M^2-q^2)}c$$

$$c = f_B M_B\left[\frac{Q_u}{\bar{\Lambda}} + \mathcal{O}\left(\frac{1}{M_B}\right)\right]$$

Ward Identities, determination of form factors and coupling constants

Define

$$\langle \gamma(k,\epsilon)|\bar{u} i\sigma^{\mu\nu} q_\nu b|B(p)\rangle = -i\varepsilon^{\mu\nu\alpha\beta}\epsilon^*_\nu k_\alpha p_\beta F_1(q^2)$$

$$\langle \gamma(k,\epsilon)|\bar{u} i\sigma^{\mu\nu}\gamma_5 q_\nu b|B(p)\rangle = [(q\cdot k)\epsilon^{*\mu} - (\epsilon^*\cdot q)k^\mu]F_3(q^2)$$

Ward Identities used to relate different form factors appearing in our calculation are

$$\begin{aligned}\langle \gamma(k,\epsilon)|\bar{u} i\sigma^{\mu\nu} q_\nu b|B(p)\rangle &= -(m_b+m_q)\langle \gamma(k,\epsilon)|\bar{u}\gamma^\mu b|B(p)\rangle \\ &\quad +(p^\mu+k^\mu)\langle \gamma(k,\epsilon)|\bar{u} b|B(p)\rangle \\ &= -(m_b+m_q)\langle \gamma(k,\epsilon)|\bar{u}\gamma^\mu b|B(p)\rangle\end{aligned}$$

$$\begin{aligned}\langle \gamma(k,\epsilon)|\bar{u} i\sigma^{\mu\nu}\gamma_5 q_\nu b|B(p)\rangle &= (m_b-m_q)\langle \gamma(k,\epsilon)|\bar{u}\gamma^\mu\gamma_5 b|B(p)\rangle \\ &\quad +(p^\mu+k^\mu)\langle \gamma(k,\epsilon)|\bar{u}\gamma_5 b|B(p)\rangle \\ &= (m_b-m_q)\langle \gamma(k,\epsilon)|\bar{u}\gamma^\mu\gamma_5 b|B(p)\rangle\end{aligned}$$

Using gauge invariance we have

$$F_V(q^2) = \frac{1}{m_b + m_q} F_1(q^2)$$

$$F_A(q^2) = \frac{1}{m_b - m_q} F_3(q^2)$$

To make use of Ward Identities to relate different form factors, define

$$\langle \gamma(k,\epsilon)|i\bar{u}\sigma_{\alpha\beta}b|B(p)\rangle = -i\varepsilon_{\alpha\beta\rho\sigma}\epsilon^{*\rho}(k)[(p+k)^\sigma g_+ + q^\sigma g_-]$$

$$-iq\cdot\epsilon^*(k)\varepsilon_{\alpha\beta\rho\sigma}(p+k)^\rho q^\sigma h$$

$$-i[q_\alpha\varepsilon_{\beta\rho\sigma\tau}\epsilon^{*\rho}(k)(p+k)^\sigma q^\tau - \alpha\leftrightarrow\beta]h_1$$

$$-i[(p+k)_\alpha\varepsilon_{\beta\rho\sigma\tau}\epsilon^{*\rho}(k)(p+k)^\sigma q^\tau - \alpha\leftrightarrow\beta]h_2$$

And using Dirac algebra we can write

$$\langle \gamma(k,\epsilon)|i\bar{u}\sigma^{\mu\nu}\gamma_5 b|B(p)\rangle = -\frac{i}{2}\varepsilon^{\mu\nu\alpha\beta}\langle \gamma(k,\epsilon)|i\bar{u}\sigma_{\alpha\beta}b|B(p)\rangle$$

Using Gauge Invariance we can write

$$F_1(q^2) = 2[g_+ - q^2 h_1 - M_B^2 h_2]$$

$$F_3(q^2) = 2\left[-g_+ - q^2 h - \left(M_B^2 - q^2\right) h_2\right]$$

Finally

$$F_V\left(q^2\right) = \frac{2}{m_b + m_q}\left\{g_+\left(q^2\right) - q^2 h_1\left(q^2\right) - M_B^2 h_2\left(q^2\right)\right\}$$

$$F_A\left(q^2\right) = \frac{2}{m_b - m_q}\left\{g_+\left(q^2\right) - q^2 h\left(q^2\right) - \left(M_B^2 - q^2\right) h_2\left(q^2\right)\right\}$$

The normalization of these form factors at $q^2 = 0$ is determined by the universal from factor $g_+(0)$ and it gets the contribution from quark-triangle graph,

$$g_+(q^2) = f_B\left\{\frac{Q_u}{2\bar{\Lambda}} - \frac{Q_b}{2M_B}\left(1 - \frac{m_q}{M_B}\right)\right\}\frac{1}{1-q^2/M_B^2}$$

$$(m_b + m_q)F_V(0) = 2g_+(0) = (m_b - m_q)F_A(0)$$

$$R\left(1 - \frac{M^2}{M_2^2}\right) + R_1\left(1 - \frac{M_1^2}{M_2^2}\right) = \left(\frac{2g_+(0)}{M}\right)\frac{M^2}{M_2^2}$$

Restricting to the one radial excitation we have

$$R = \frac{2g_+(0)}{M\left(M_1^2/M^2-1\right)}$$

$$F(q^2) = \frac{2}{M}\frac{g_+(0)}{\left(1-q^2/M^2\right)\left(1-q^2/M_1^2\right)}$$

So the coupling constants become

$$g_{B^*B\gamma} = \frac{2g_+(0)}{M_B} \frac{M_{B^*}^2}{f_{B^*}\left(M_{B_1^*}^2/M_{B^*}^2-1\right)}$$

$$f_{B_A^*B\gamma} = \frac{M_{B_A^*}^2}{M_B} \frac{2g_+(0)}{f_{B_A^*}\left(M_{B_{A_1}^*}^2/M_{B_A^*}^2-1\right)}$$

Using

$$g_+(0) = \frac{2}{3}\frac{f_B}{2\bar{\Lambda}}$$

We have the prediction

$$g_{B^*B\gamma} = \frac{2}{3\bar{\Lambda}} \frac{1}{\left(M_{B_1^*}^2/M_{B^*}^2-1\right)}$$

The final form of the form factors become

$$F_V(q^2) = \frac{2}{M_B} \frac{g_+(0)}{\left(1-q^2/M_{B^*}^2\right)\left(1-q^2/M_{B_1^*}^2\right)}$$

$$F_A(q^2) = \frac{2}{M_B} \frac{g_+(0)}{\left(1-q^2/M_{B_A^*}^2\right)\left(1-q^2/M_{B_{A_1}^*}^2\right)}$$

Using the numerical values of the parameters involved we have

$$g_{B^*B\gamma} = \frac{2.2}{\bar{\Lambda}} = 5.6\ \text{GeV}^{-1} \qquad g_+(0) = \frac{3}{20} = 0.15$$

$$f_{B_A^*B\gamma} = 6.5\frac{f_B M_{B_A^*}}{f_{B_A^*}}\ \text{GeV}^{-1}$$

Second Radial Excitation

$$F\left(q^2\right) = \frac{R\left(\frac{M_2^2}{M^2}-1\right)\left(\frac{M_1^2}{M^2}-1\right)\frac{M^2}{M_2^2}\frac{q^2}{M_1^2}+\frac{2g_+(0)}{M}\left(1-q^2\left(\frac{1}{M_2^2}+\frac{1}{M_1^2}-\frac{M^2}{M_1^2M_2^2}\right)\right)}{\left(1-q^2/M_2^2\right)\left(1-q^2/M_1^2\right)\left(1-q^2/M^2\right)}$$

$$R = \frac{2g_+(0)}{M}\frac{1-\left(1-M_1^2/M_2^2\right)A}{\left(M_1^2/M^2-1\right)}$$

where A is the parameter which in principle can be obtained when $g_{B^*B\gamma}$ and $f_{B_A^*B\gamma}$ become known. Then

$$F\left(q^2\right) = \frac{2g_+(0)}{M}\frac{1-\frac{q^2}{M_1^2}\left(1+\left(1-\frac{M^2}{M_2^2}\right)\left(1-\frac{M_1^2}{M_2^2}\right)A\right)}{\left(1-q^2/M_2^2\right)\left(1-q^2/M_1^2\right)\left(1-q^2/M^2\right)}$$

For numerical analysis we use $A = 0, 3.0, 4.8$. Now the coupling of B with $B^*\gamma$ and $B_A^*\gamma$ becomes

$$g_{B^*B\gamma} = \left[1 - \left(1 - M_{B^*}^2/M_{B_1^*}^2\right)\left(1 - M_{B_1^*}^2/M_{B^*}^2\right)A\right] 5.6 \text{ GeV}^{-1}$$

$$f_{B_A^*B\gamma} = \frac{f_B M_{B_A^*}}{f_{B_A^*}}\left[1 - \left(1 - M_{B_A^*}^2/M_{B_{A_1}^*}^2\right)\left(1 - M_{B_{A_1}^*}^2/M_{B_A^*}^2\right)A\right] 6.5 \text{ GeV}^{-1}$$

The corresponding form factors becomes

$$F_V(q^2) = \frac{2g_+(0)}{M_B}\frac{1 - \frac{q^2}{M_{B_1^*}^2}\left(1 + \left(1 - \frac{M_{B^*}^2}{M_{B_2^*}^2}\right)\left(1 - \frac{M_{B_1^*}^2}{M_{B_2^*}^2}\right)A\right)}{\left(1 - q^2/M_{B_2^*}^2\right)\left(1 - q^2/M_{B_1^*}^2\right)\left(1 - q^2/M_{B^*}^2\right)}$$

$$F_A(q^2) = \frac{2g_+(0)}{M_B}\frac{1 - \frac{q^2}{M_{B_{A_1}^*}^2}\left(1 + \left(1 - \frac{M_{B_A^*}^2}{M_{B_{A_2}^*}^2}\right)\left(1 - \frac{M_{B_{A_1}^*}^2}{M_{B_{A_2}^*}^2}\right)A\right)}{\left(1 - q^2/M_{B_{A_2}^*}^2\right)\left(1 - q^2/M_{B_{A_1}^*}^2\right)\left(1 - q^2/M_{B_A^*}^2\right)}$$

For numerical values we shall use $A = 0$, $A = 3$ and $A = 4.8$. The second value of $A(= 3)$ corresponds to estimate of $g_{B^*B\gamma}$ from vector meson dominance

$$g_{B^*B\gamma} = \frac{2}{3} g_{B^*B\rho^-} \frac{f_{\rho^-}}{m_\rho^2} = 2.76 \text{ GeV}^{-1}$$

$$g_{B^*B\rho^-} = \sqrt{2}(11) \text{ GeV}^{-1} \ , \ f_{\rho^-}/m_\rho = 205 \text{ MeV}$$

The third value of A(=4.8) gives more or less the width for $B^* \to B\gamma$ obtained from MI transition in non relativistic quark model (NRQM). These values give decay width for $B^* \to B\gamma$ transition 23 keV, 5.5 keV and 0.8 keV respectively while MI transition in NRQM predicts it to be 0.9 keV. These predictions are testable when above decay width is experimentally measured.

The final expression for form factors in terms of the dimensionless variable x becomes

$$F_V(x) = \frac{F_V(0)}{x\left[1-(1-x)/\left(M_{B_1^*}/M_{B^*}\right)^2\right]}$$

$$F_A(x) = \frac{F_V(0)}{x\left[1-(1-x)/\left(M_{B^*_{A_1}}/M_{B^*_A}\right)^2\right]}$$

$$F_{V,A}(0) = \frac{2g_+(0)}{M_B}$$

Branching Ratio

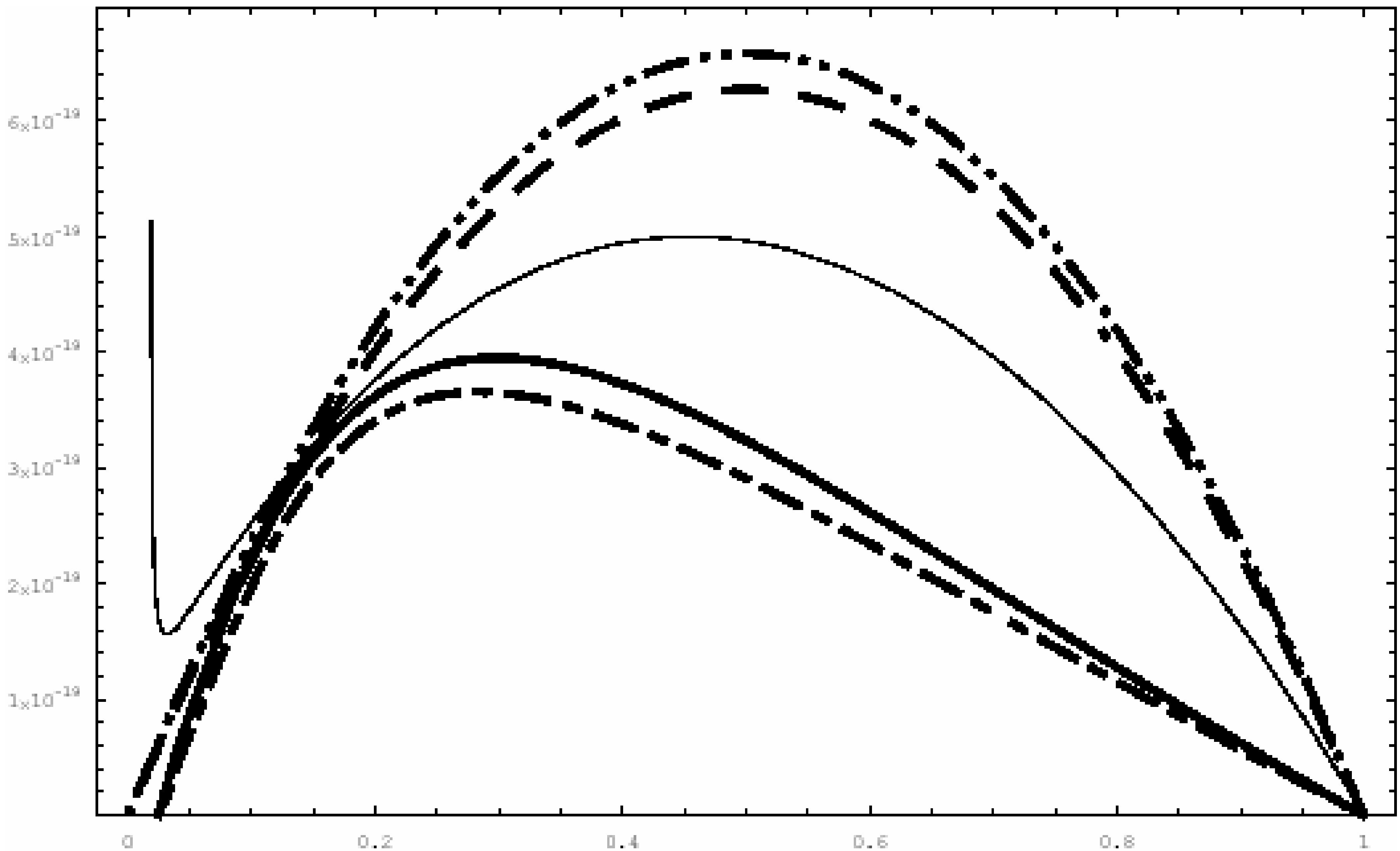
Using the form factors calculated above we have

$$\mathcal{B}(B \to \gamma l \nu_l) = 0.5 \times 10^{-6} \qquad for \;\; l = \mu$$

$$= 0.38 \times 10^{-6} \quad (l = \mu, A = 3.0)$$

$$= 0.32 \times 10^{-6} \quad (l = \mu, A = 4.8)$$

- CLEO 2×10^{-6}
- Bethe-Salpeter approach 0.9×10^{-6}
- Light-Cone QCD $(2\text{-}5) \times 10^{-6}$
- Monte-Carlo Simulation 5.2×10^{-5}



6x10^-19
5x10^-19
4x10^-19
3x10^-19
2x10^-19
1x10^-19
0
0.2
0.4
0.6
0.8
1

Conclusion

- We have studied $B \rightarrow \gamma l \nu_l$ decay using dispersion relations, asymptotic behavior and Ward Identities.
- The dispersion relation involves ground state B^* and B_A^* resonances and their radial excitations which model contributions from higher states and continuum contribution, which is calculated from quark triangle graph.
- The asymptotic behavior of form factors and Ward Identities fix the normalization of the form factors in terms of universal function $g_+(0)$ at $q^2=0$ and put constraints on the residues. Thus in our approach, a parameterization of q^2 dependence of form factors is not approximated by single pole contributions.
- Taking into account the radial excitations, form factors and coupling constants are calculated. Using these as an input Branching ratio is calculated and compared it with different approaches.
- Finally the partial decay width vs. the photon energy spectrum is plotted and it is found that our peak shifts towards the lower value of x.

Thanks!