Topological Strings and Special Holonomy Manifolds

Asad Naqvi

(University of Wales, Swansea)

with Jan de Boer and Assaf Shomer

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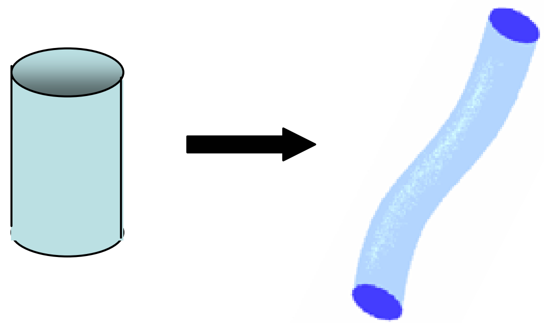
String Theory

X^μ : 

We introduce an action, which is simply the area of the world sheet.

$$S = \frac{1}{2\pi l_s^2} \int d^2z \; G_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu$$

To find the probability of transition, we follow Feynman's prescription of integrating over all possible paths between two points.

$$A_{1\to 2} = \int [dX^\mu]_{1,2} \; e^{-S} = \int [dX^\mu] e^{-S} \mathcal{O}_1 \mathcal{O}_2$$

Constructing Topological Strings

Superstrings on M

$X^\mu : \Sigma \to M$

$S[X^\mu, \psi^\mu]$

Twist →

$S[X^\mu, \psi^\mu] + \delta S$

Has SUSY Q which is a spinor on the world sheet

Has SUSY Q which is a scalar on the world sheet

Now consider only those operators $\{\mathcal{O}_i\}$ which are in the cohomology of Q

$$\{Q, \mathcal{O} = 0\} \qquad \mathcal{O} \sim \mathcal{O} + \{Q, \phi\}$$

If *M* is a Calabi-Yau manifold, then there are 2 ways we can twist, the A and the B twist

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \int [dX^\mu] \mathcal{O}_1 \cdots \mathcal{O}_n \, e^{-S_{twisted}}$$

A twist: Holomorphic maps

B twist: Constant maps

Also, $\partial_\mu \mathcal{O} = \{Q, \phi\}$

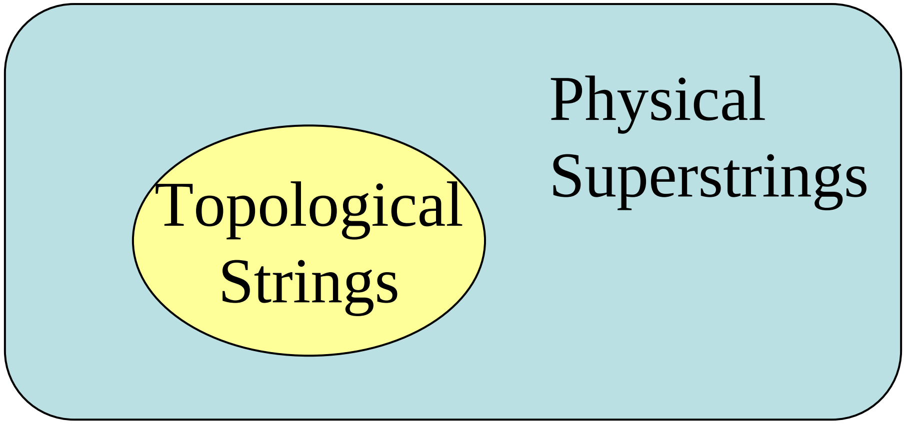
Hence these correlation functions are position independent. They are functions of the moduli of the Calabi-Yau.

- A-model only depends on the Kahler structure moduli
- B-model only depends on the Complex structure moduli

Topological strings have provided a useful insights into various physical and mathematical questions

- They are useful toy models of string theories which are still complicated enough to exhibit interesting phyiscal phenomena in a more controlled setting

- The describe a sector of superstrings and provide exact answers to certain questions concerning BPS quantities



Outline

- G_2 manifolds
- G_2 sigma models

  (1,1) SUSY ⇒ Extended symmetry algebra

- Tricritical Ising model algebra is contained in this extended algebra **Shatashvili and Vafa 9407025**
- Topological twist of the G_2 sigma model
- Relation to Geometry

G_2 manifolds

Special holonomy $G_2 \subset SO(7)$

Under this embedding $8 \rightarrow 1 \oplus 7$

i.e. there is a covariantly constant spinor

$$\nabla \epsilon = 0$$

⇓

$$\phi(p) = \epsilon^T \Gamma_{i_1 \dots i_p} \epsilon dx^{i_1} \wedge \dots \wedge dx^{i_p}$$

is a covariantly constant p-form

This is non-zero for p=0,3,4 and 7

G_2 sigma models

Lets start with a (1,1) sigma model

$$S = \int d^2z \; d^2\theta \; (G_{\mu\nu} + B_{\mu\nu}) D_\theta \mathbf{X}^\mu D_{\bar\theta} \mathbf{X}^\nu$$

where

$$D_\theta = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z} \qquad , \qquad D_{\bar\theta} = \frac{\partial}{\partial\bar\theta} + \bar\theta\frac{\partial}{\partial\bar z}$$

$$\mathbf{X}^\mu = \phi^\mu(z) + \theta\psi^\mu(z)$$

This model has (1,1) supersymmetry

$$\mathbf{T}(z,\theta) = G(z) + \theta T(z) = -\frac{1}{2} G_{\mu\nu} D_\theta \mathbf{X}^\mu \partial_z \mathbf{X}^\nu$$

G-structures and Extended Chiral Algebra

Covariantly constant forms → Extra holomorphic currents

Given a covariantly constant p-form satisfying

$$\nabla \phi_{i_1 \cdots i_p} = 0$$

the current

$$\mathbf{J}_{(p)}(z,\theta) = \phi_{i_1 \cdots i_p} D_\theta \mathbf{X}^{i_1} \cdots D_\theta \mathbf{X}^{i_p}$$

satisfies

$$D_{\bar{\theta}} \mathbf{J}_{(p)} = 0$$

↓

dim $\frac{p}{2}$ and dim $\frac{p+1}{2}$ currents

Kahler manifolds-an example

On a Kahler manifold, a Kahler form

$$\omega = g_{i\bar{j}}(d\phi^i \wedge d\phi^{\bar{j}} - d\phi^{\bar{j}} \wedge d\phi^i)$$

implies the existence of a dimension 1 current

$$J = g_{i\bar{j}}\psi^i\psi^{\bar{j}}$$

and a dimension $\frac{3}{2}$ current

$$G'(z) = g_{i\bar{j}}(\psi^i\partial_z\phi^{\bar{j}} - \psi^{\bar{j}}\partial_z\phi^i)$$

which extend the (1,1) algebra to a (2,2) algebra

Extended G_2 algebra

A G_2 holonomy manifold has a covariantly constant 3-form

$$\phi^{(3)} = \phi^{(3)}_{ijk} dx^i \wedge dx^j \wedge dx^k$$

which implies the existence of

$$\mathbf{J}_{(3)}(z,\theta) = \phi^{(3)}_{ijk} D_\theta \mathbf{X}^i D_\theta \mathbf{X}^j D_\theta \mathbf{X}^k \equiv \Phi + \theta K$$

where $\Phi = \phi^{(3)}_{ijk} \psi^i \psi^j \psi^k$ and $K = \phi^{(3)}_{ijk} \psi^i \psi^j \partial \phi^k$

There is also a covariantly constant 4 form which leads to a dimension 2 current X and a dimension 5/2 current M

$h = \frac{3}{2}$	$G(z)$	$\Phi(z)$	
$h = 2$	$T(z)$	$K(z)$	$X(z)$
$h = \frac{5}{2}$			$M(z)$

OPEs

$$\Phi(z)\Phi(w) = -\frac{7}{(z-w)^3} + \frac{6}{(z-w)}X(w)$$

$$\Phi(z)X(w) = -\frac{15}{2(z-w)^2}\Phi(w) - \frac{5}{2(z-w)}\partial\Phi(w)$$

$$X(z)X(w) = \frac{35}{4(z-w)^4} - \frac{10}{(z-w)^2}X(w) - \frac{5}{(z-w)}\partial X(w)$$

$\mathcal{N}=1$
SCA
with
$c = \frac{7}{10}$
$T_I = -\frac{1}{5}X$
$G_I = \frac{i}{\sqrt{15}}\Phi$

$$G(z)\Phi(w) = \frac{1}{z-w}K(w)$$

$$G(z)X(w) = -\frac{1}{2(z-w)^2}G(w) + \frac{1}{z-w}M(w)$$

$$\Phi(z)M(w) = \frac{9}{2}\frac{1}{(z-w)^2}K(w) - \frac{1}{z-w}(3 : G(w)\Phi(w) : -\frac{5}{2}\partial K(w)$$

$$X(z)K(w) = -\frac{3}{(z-w)^2}K(w) + \frac{3}{z-w}\Big(: G(w)\Phi(w) : -\partial K(w)\Big)$$

⋮

An important fact is that

$$T_I(z)T_r(w) = 0 \quad , \qquad\qquad T = T_I + T_r$$

which means that states of the CFT can be labeled by its tri-critical Ising model weight and its weight in the remainder

$$|\varphi\rangle = |h_I, h_r\rangle$$

Tricritical Ising Model

$$c = \frac{7}{10}$$

Kac table: Spectrum of conformal primaries

$n' \backslash n$	1	2	3
1	0	$\frac{7}{16}$	$\frac{3}{2}$
2	$\frac{1}{10}$	$\frac{3}{80}$	$\frac{6}{10}$
3	$\frac{6}{10}$	$\frac{3}{80}$	$\frac{1}{10}$
4	$\frac{3}{2}$	$\frac{7}{16}$	0

Some fusion rules:-

$$\phi_{1,2} \times \phi_{n',n} = \phi_{n',n-1} + \phi_{n',n+1}$$
$$\phi_{2,1} \times \phi_{n',n} = \phi_{n'-1,n} + \phi_{n'+1,n}$$

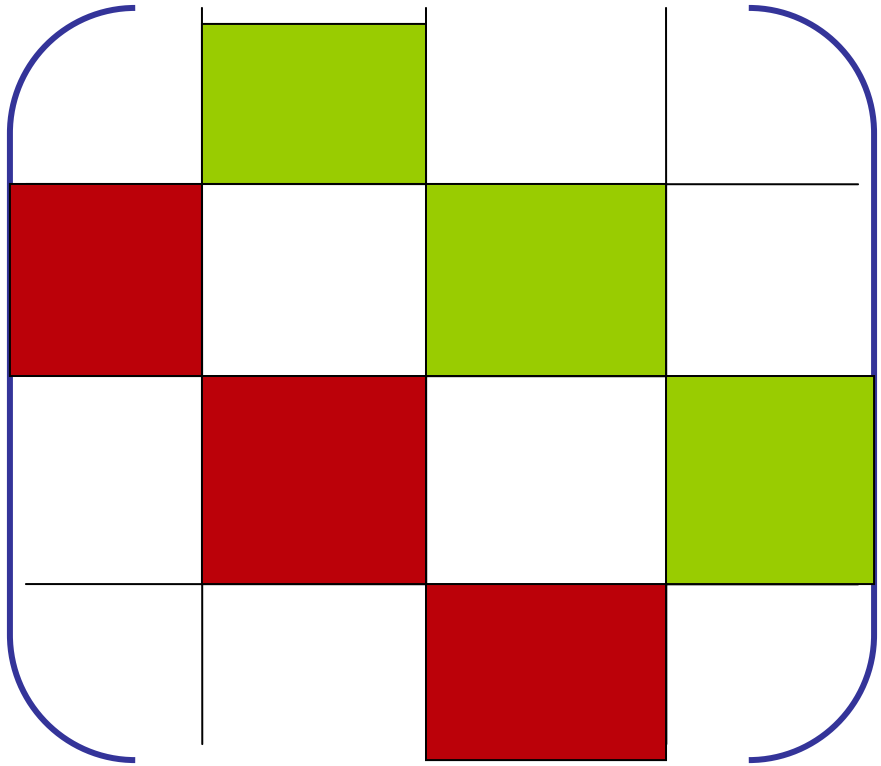
$$\frac{7}{16} \times \frac{3}{80} = \frac{1}{10} + \frac{6}{10}$$

$$\frac{1}{10} \times \frac{1}{10} = 0 + \frac{6}{10}$$
$$\frac{1}{10} \times \frac{6}{10} = \frac{1}{10} + \frac{3}{2}$$
$$\frac{1}{10} \times \frac{3}{2} = \frac{6}{10}$$

Conformal blocks

The fusion rules imply

$$\phi_{2,1} \sim$$



$$\equiv \phi^{\uparrow}_{2,1} + \phi^{\downarrow}_{2,1}$$

$$\Downarrow$$

$$h = \frac{1}{10}$$

A Unitarity Bound

$$h_I + h_r \geq \frac{1 + \sqrt{1 + 80 h_I}}{8}$$

States which saturate the bound will be called chiral primary

Notice the definition of chiral primaries involve a non-linear inequality.

We will see later that the topological theory keeps only the chiral primary states

$$|\frac{1}{10}, \frac{2}{5}\rangle$$
$$|\frac{6}{10}, \frac{2}{5}\rangle$$
$$|\frac{3}{2}, 0\rangle$$

$|\frac{1}{10},\frac{2}{5}\rangle$ has dimension $\frac{1}{2}$

So $G_{-\frac{1}{2}}|\frac{1}{10},\frac{2}{5}\rangle$ preserves $\mathcal{N}=1$ and is dim 1

$$\mathcal{M} \sim |\frac{1}{10},\frac{2}{5}\rangle_L \times |\frac{1}{10},\frac{2}{5}\rangle_R$$

$$\Delta S = \int d^2z\, G_{-\frac{1}{2}}\bar{G}_{-\frac{1}{2}}\mathcal{M}$$

Shatashvili+Vafa
1994

is a candidate for an exactly marginal deformation

Ramond Sector

Ramond sector ground states: dim = $\frac{7}{16}$

$$|\frac{7}{16},0\rangle \quad ; \quad |\frac{3}{80},\frac{2}{5}\rangle$$

Topological Twist

Review of the Calabi-Yau twisting

Sigma model action:-

$$\int d^2z\{\frac{1}{2}g_{i\bar{j}}(\partial x^i\bar{\partial}x^{\bar{i}}+\partial x^{\bar{i}}\bar{\partial}x^i)$$
$$+g_{i\bar{j}}(i\psi_-^{\bar{j}}D\psi_-^i+i\psi_+^{\bar{j}}\bar{D}\psi_+^i)+R_{i\bar{j}k\bar{l}}\psi_+^i\psi_+^{\bar{j}}\psi_-^k\psi_-^{\bar{l}}\}$$

B-twist A-twist **1-form** **scalar** **scalar 1-form** **1-form scalar**

Effectively, we are adding background gauge field for the U(1)

$$D=\partial+\frac{\omega}{2}\rightarrow D'=\partial+\frac{\omega}{2}+eA$$

with

$$A=\frac{\omega}{2}$$

$$\delta S = \int g_{i\bar{j}} \psi_+^i \psi_+^{\bar{j}} \ \frac{\bar{\omega}}{2} = -i\frac{\sqrt{d}}{2}\int \phi \partial \bar{\omega} = -i\frac{\sqrt{d}}{2}\int \phi R$$

$$i\sqrt{d}\partial\phi$$

So on a sphere

$$e^{-\delta S} = e^{i\frac{\sqrt{d}}{2}\phi(0)} e^{i\frac{\sqrt{d}}{2}\phi(\infty)}$$

Since

$$\langle \cdots \rangle = \int \mathcal{D}x\mathcal{D}\psi^i \cdots e^{-S} e^{-\delta S}$$

$$\langle \cdots \rangle_{\text{twisted}} = \langle e^{i\frac{\sqrt{d}}{2}\phi(\infty)} \ldots e^{i\frac{\sqrt{d}}{2}\phi(0)} \rangle_{\text{untwisted}}$$

On higher genus surfaces, we need 2-2g insertions

This effectively adds a background charge $Q = \sqrt{d}$ for the U(1) part thereby changing its central charge.

$$c = \frac{3}{2} \times 2d \to 1 - 3Q^2 + 3d - 1 = 0$$

Twisting the G_2 sigma model

We apply this to the G_2 sigma model

The role of $e^{i\frac{\sqrt{d}}{2}\phi}$ will be played by $|\frac{7}{16},0\rangle$

$e^{i\frac{\sqrt{d}}{2}\phi}$ sits purely within the $U(1)=\frac{U(d)}{SU(d)}$

For the G_2 sigma model the role of the U(1) part is played by the tri-critical Ising model

$$c(\frac{SO(7)}{G_2})=\frac{7}{10}$$

Back to the G_2 twist

Correlation functions

$$\langle O_1(z_1)) \cdots O_k(z_k) \rangle_{\text{twisted}} = \langle V^{+}_{\frac{7}{16}}(\infty) O_1(z_1)) \cdots O_k(z_k) V^{+}_{\frac{7}{16}}(0) \rangle_{\text{untwisted}}$$

BRST Operator (Scalar Q)

We can show that $G(z) = \Phi_{2,1} \otimes \psi_{\frac{14}{10}}$

This splits as $G = G^{\uparrow} + G^{\downarrow}$

$$Q = \oint G^{\downarrow}$$

Projectors

As we saw before, a generic state in the theory can be labeled by two qunatum numbers:-

$$|\varphi\rangle = |h_I, h_r\rangle$$

h_I is the weight of the state under the tri-critical Ising part. For primary fields

$$h_I = \Delta(k) = \frac{2k^2 - k}{10}$$
$$= 0\ ,\ \frac{1}{10},\ \ \frac{6}{10},\ \frac{3}{2}$$

Define P_k to be the projector which projects onto the k^{th} conformal family

$$P_0 + P_1 + P_2 + P_3 = 1$$

BRST and its Cohomology

The BRST operator that can be written as

$$Q = \sum_k P_{k+1} G_{-1/2} P_k$$

This squares to zero:-

$$Q^2 = \sum_k P_{k+2} G^2_{-1/2} P_k = \sum_k P_{k+2} L_{-1} P_k = 0$$

State Cohomology

From the tri-critical fusion rules, we know that

$$G_{-1/2}|\Delta(k), h_r\rangle = c_1|\Delta(k-1), h_r - \Delta(k-1) + \Delta(k) + \frac{1}{2}\rangle + c_2|\Delta(k+1), h_r - \Delta(k+1) + \Delta(k) + \frac{1}{2}\rangle$$

Then, by definition

$$Q|\Delta(k), h_r\rangle = c_2|\Delta(k+1), h_r - \Delta(k+1) + \Delta(k) + \frac{1}{2}\rangle$$

$$\langle \Delta(k), h_r | G_{1/2} G_{-1/2} | \Delta(k), h_r \rangle = 2(\Delta(k) + h_r) = |c_1|^2 + |c_2|^2$$

$$\{G_{1/2}, G_{-1/2}\} = 2L_0$$

$$\begin{aligned}\langle \Delta(k), h_r | G_{1/2} X_0 G_{-1/2} | \Delta(k), h_r \rangle &= 9\Delta(k) - h_r - 10\Delta(k)(\Delta(k) + h_r) \\ &= -5\Delta(k-1)|c_1|^2 - 5\Delta(k+1)|c_2|^2\end{aligned}$$

We can solve for c_1 and c_2 upto an irrelevant phase and $c_2=0$ implies

$$\Delta(k) + h_r = \frac{10\Delta(k)}{10\Delta(k) + 1 - 10\Delta(k-1)} = \frac{k}{2} = \frac{1 + \sqrt{1 + 80\Delta(k)}}{2}$$

This is precisely the unitarity bound that we found earlier.

Dolbeault Cohomology for G_2 and the chiral BRST Cohomology

For a G_2 manifold, forms at each degree can be decomposed in irreducible representations of G_2.

$$\Lambda^0 = \Lambda^0_1 \qquad \Lambda^1 = \Lambda^1_7$$

$$\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{14} \qquad \Lambda^3 = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$$

Cohomology groups decompose as $H^p_R(M)$ and depend on the G_2 irrep *R* only and not on *p*

We can define a sub-complex of the de Rham complex as follows

$$0 \longrightarrow \Lambda^0_1 \xrightarrow{\check{D}} \Lambda^1_7 \xrightarrow{\check{D}} \Lambda^2_7 \xrightarrow{\check{D}} \Lambda^3_1 \longrightarrow 0$$

We will next see that this $\check{D}$ operator maps to our BRST operator Q

BRST Cohomology Geometrically

$$\omega_{i_1,\ldots,i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p} \leftrightarrow \omega(x^\mu)_{i_1,\ldots,i_p} \psi^{i_1} \ldots \psi^{i_p}$$

The following table summarizes the L_0 and X_0 eigenvalues of these operators

	1	**7**	**14**	**27**
$p=0$	$\|0,0\rangle$			
$p=1$		$\|\frac{1}{10},\frac{2}{5}\rangle$		
$p=2$		$\|\frac{6}{10},\frac{2}{5}\rangle$	$\|0,1\rangle$	
$p=3$	$\|\frac{3}{2},0\rangle$	$\|\frac{11}{10},\frac{2}{5}\rangle$		$\|\frac{1}{10},\frac{7}{5}\rangle$
$p=4$	$\|2,0\rangle$	$\|\frac{16}{10},\frac{2}{5}\rangle$		$\|\frac{6}{10},\frac{7}{5}\rangle$
$p=5$		$\|\frac{21}{10},\frac{2}{5}\rangle$	$\|\frac{3}{2},1\rangle$	
$p=6$		$\|\frac{26}{10},\frac{2}{5}\rangle$		
$p=7$	$\|\frac{7}{2},0\rangle$			

Differential Complexes

	1	**7**	**14**	**27**
$p=0$	$\lvert 0,0\rangle$			
$p=1$		$\lvert \frac{1}{10},\frac{2}{5}\rangle$		
$p=2$		$\lvert \frac{6}{10},\frac{2}{5}\rangle$	$\lvert 0,1\rangle$	
$p=3$	$\lvert \frac{3}{2},0\rangle$	$\lvert \frac{11}{10},\frac{2}{5}\rangle$		$\lvert \frac{1}{10},\frac{7}{5}\rangle$
$p=4$	$\lvert 2,0\rangle$	$\lvert \frac{16}{10},\frac{2}{5}\rangle$		$\lvert \frac{6}{10},\frac{7}{5}\rangle$
$p=5$		$\lvert \frac{21}{10},\frac{2}{5}\rangle$	$\lvert \frac{3}{2},1\rangle$	
$p=6$		$\lvert \frac{26}{10},\frac{2}{5}\rangle$		
$p=7$	$\lvert \frac{7}{2},0\rangle$			

$$0 \longrightarrow \Lambda^2_{14} \xrightarrow{\tilde{D}} \Lambda^3_7 \oplus \Lambda^3_{27} \xrightarrow{\tilde{D}} \Lambda^4_7 \oplus \Lambda^4_{27} \xrightarrow{\tilde{D}} \Lambda^5_{14} \longrightarrow 0$$

$$G_{-1/2}A_\mu(X)\psi^\mu = \frac{1}{2}\partial_{[\nu}A_{\mu]}\psi^\nu\psi^\mu + A_\mu(X)\partial X^\mu$$

⇩ **7 + 14** ⇩ 1

Projection operator onto the 7 when acting on 2 forms is

$$P_{ab}{}^{de} = 6\phi_{ab}{}^c\phi_c{}^{de}$$

$$G^{\downarrow}_{-1/2}A_\mu(X)\psi^\mu = 3\partial_{[\nu}A_{\mu]}\phi^{\nu\mu}{}_\rho\phi^\rho{}_{\alpha\beta}\psi^\alpha\psi^\beta$$

We can repeat this analysis for the two and three forms

Chiral BRST Cohomology

$$1$$

$$A_\mu \psi^\mu \quad \text{with} \quad \phi_\rho{}^{\mu\nu} \partial_{[\mu} A_{\nu]} = 0$$

$$B_{\mu\nu} \psi^\mu \psi^\nu \quad \text{with} \quad \phi^{\rho\mu\nu} \partial_{[\rho} B_{\mu\nu]} = 0$$

$$\phi_{\mu\nu\rho} \psi^\mu \psi^\nu \psi^\rho .$$

with

$$A_\mu \sim A_\mu + \partial_\mu C$$

$$B_{\alpha\beta} \sim B_{\alpha\beta} + 3\partial_{[\nu} D_{\mu]} \phi^{\nu\mu}{}_\rho \phi^\rho{}_{\alpha\beta}$$

This is exactly the cohomology of the $\check{D}$ operator

Almost trivial since $b^1_{\mathbf{7}} = 0$

Total BRST Cohomology

If we combine the left movers with the right movers, we get a more interesting cohomology

$$\mathbf{7} \times \mathbf{7} = \mathbf{1} + \mathbf{7} + \mathbf{14} + \mathbf{27}$$

$$\downarrow \qquad \downarrow$$

$$\mathbf{b}_2 \qquad \mathbf{b}_3 - 1$$

$$\begin{array}{lcll} 0-\mathrm{form} & \times & 0-\mathrm{form} & \longrightarrow \quad b_0 \\ 1-\mathrm{form} & \times & 1-\mathrm{form} & \longrightarrow \quad b_2 + b_3 \\ 2-\mathrm{form} & \times & 2-\mathrm{form} & \longrightarrow \quad b_4 + b_5 \\ 3-\mathrm{form} & \times & 3-\mathrm{form} & \longrightarrow \quad b_7 \end{array}$$

Full de Rham cohomology $H^*(M)$

The metric and B-field moduli should be given by operators of the form

$$(\delta g_{\mu\nu} + \delta B_{\mu\nu})\psi_R^{\mu}\psi_L^{\nu}$$

with

$${\phi_\alpha}^{\lambda\mu}(\nabla_{[\lambda}\delta g_{\mu]\nu} + \nabla_{[\lambda}\delta B_{\mu]\nu}) = 0$$

Correlation Functions

Consider three point function of operators

$$Y = \mathcal{O}_{\frac{1}{10},\frac{2}{5}} \otimes \mathcal{O}_{\frac{1}{10},\frac{2}{5}}$$

$$\Downarrow$$

$$\delta g_{jk}\ \psi_L^j \psi_R^k$$

$$\langle Y_i Y_j Y_k \rangle = \int_M d^7x \sqrt{g}\,\phi_{abc} \frac{\partial g^{aa'}}{\partial t_i} \frac{\partial g^{bb'}}{\partial t_j} \frac{\partial g^{cc'}}{\partial t_k} \phi_{a'b'c'}$$

On general grounds, we expect this is the third derivative of a prepotential if suitable flat coordinates are used for the moduli space of G_2 metrics.

$$\langle Y_i Y_j Y_k \rangle = -\frac{1}{21} \frac{\partial^3}{\partial t_i \partial t_j \partial t_k} \int \phi \wedge *\phi .$$

In fact, the generating function of all our correlation functions is given by

$$\mathcal{I}_{\rm tot} = -\frac{1}{21} \int \phi \wedge *\phi - \frac{1}{216} \int B \wedge B \wedge \phi .$$

G_2 Special Geometry

Define,

$$t^A = \int_{C_A} \phi \qquad\qquad F_A = \int_{D^A} *\phi.$$

and

$$\mathcal{I} = \int \phi \wedge *\phi = t^A F_A$$

In fact,

$$F_B = \frac{3}{7}\partial_B \mathcal{I}$$

and

$$\partial_A \partial_B \partial_C \mathcal{I} = -21 \int \sqrt{g}\phi_{abc}\partial_A h^{aa'}\partial_B h^{bb'}\partial_C h^{cc'}\phi_{a'b'c'}$$

Conclusions

- We have constructed a new topological theory in 7 dimensions which captures the geometry of G_2 manifolds

- Relation to topological M-theory ?

- D-branes ?

- Spin 7 ?

⋮