Quantum Computing – New Frontiers

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Quantum Computation

Computational power could exceed that of Turing machines

History:
  Potential of quantum computer
  Simulation of quantum-mechanical objects by other quantum systems
- D. Deutsch (1985)
  Description of universal quantum computer
Hunt for something interesting for quantum computers to do

- P. Shor (1994)
  Quantum algorithm to perform efficient factorization
  Quantum search algorithm
- 1994-present
  Quantum logic gates
  Quantum lithography
  Quantum dense coding
  Quantum teleportation
  ……
Present Status

• Entanglement: 5-bit entanglement NMR

• Quantum logic gates: ion traps, cavity QED, NMR

• Shor’s algorithm: NMR (factoring of 15)

• Grover’s algorithm: NMR (2 qubit), multilevel atomic system without entanglement
Quantum Computation

Classical bits: 0, 1
Quantum bits (qubit): |0⟩, |1⟩
quantum states

Qubits in the laboratory:
Photon number state: |0⟩, |1⟩
Photon linear polarization: |V⟩, |H⟩
Electron, nuclear spin: |↓⟩, |↑⟩
Atomic energy levels: |0⟩, |1⟩
Why Quantum Computer?

(1) Qubits can form coherent superposition

$$|\Psi\rangle = C_0|0\rangle + C_1|1\rangle \quad \quad |C_0|^2 + |C_1|^2 = 1$$

(2) Qubits can form quantum entanglement

$$\frac{1}{\sqrt{2}}(|0_1,1_2\rangle + |1_1,0_2\rangle) \quad \text{or} \quad \frac{1}{\sqrt{2}}(|0_1,0_2\rangle + |1_1,1_2\rangle)$$

$$|\psi(1,2)\rangle \neq |\psi(1)\rangle |\psi(2)\rangle$$

N qubits can store $2^N$ numbers “simultaneously”

For 2 qubits we can have

$$C_0|0,0\rangle + C_1|0,1\rangle + C_2|1,0\rangle + C_3|1,1\rangle$$
• Two particles: $$\frac{1}{2}(C_0\ket{0,0} + C_1\ket{0,1} + C_2\ket{1,0} + C_3\ket{1,1})$$

• Four particles:

$$\frac{1}{4}(C_0\ket{0,0,0,0} + C_1\ket{0,0,0,1} + C_2\ket{0,0,1,0} + C_3\ket{0,0,1,1}$$

$$+ C_4\ket{0,1,0,0} + C_5\ket{0,1,0,1} + \ldots + C_{15}\ket{1,1,1,1})$$

• N = 1000 particles:

$$0 \ 1 \ 2 \ \ldots \ 2^{1000} > 10^{256}$$

$$10^{256} \sim \text{number of atoms in the universe}$$
Problems

• For coherent superposition

\[ |\psi\rangle = C_0 |0\rangle + C_1 |1\rangle \]

experimental outcome is probabilistic

Probability of finding \( |0\rangle \) is \( |C_0|^2 \)
Probability of finding \( |1\rangle \) is \( |C_1|^2 \)

• Decoherence—any interaction with surroundings will destroy coherent superposition and entanglement
Conditions for a Quantum Computing System

• Be a scalable physical system with well defined qubits
• Be initializable to a simple state such as $|000\ldots\rangle$
• Have much longer decoherence times (i.e., one can do many operations before losing quantum coherence)
• Have a universal set of quantum gates
• Permit high quantum efficiency, qubit specific measurements
Basic building blocks:

Quantum logic gates (QLG)

Due to coherent nature of quantum mechanics, quantum logic gates should be reversible

Classical gates:

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Not reversible !!
Basic Building Blocks

Quantum logic gates (QLG)

Due to coherent nature of quantum mechanics, quantum logic gates should be reversible

One-bit gate:

\[ U_\theta |A\rangle \rightarrow U_\theta |A\rangle \]

\[ U_\theta |0\rangle = \cos(\theta / 2) |0\rangle + \sin(\theta / 2) |1\rangle \]

\[ U_\theta |1\rangle = \cos(\theta / 2) |1\rangle - \sin(\theta / 2) |0\rangle \]

Quantum phase gate:

\[ Q_\phi \]

\[ Q_\phi |a,b\rangle = e^{i\phi \delta_{a,1}\delta_{b,1}} |a,b\rangle \]

\[ \begin{cases} |0,0\rangle \rightarrow |0,0\rangle, & |0,1\rangle \rightarrow |0,1\rangle \\ |1,0\rangle \rightarrow |1,0\rangle, & |1,1\rangle \rightarrow e^{i\phi} |1,1\rangle \end{cases} \]
Quantum Phase Gate \(^a\)

- First qubit: Atomic state
  \[ |c\rangle = |0\rangle, \quad |b\rangle = |1\rangle \]

- Second qubit: Cavity photon number state
  \[ |0\rangle, \quad |1\rangle \]

- Cavity field is nearly resonant with \(|a\rangle - |b\rangle\) transition (Dispersive coupling)
Quantum Phase Gate

• Effective Hamiltonian:

\[ H_{\text{eff}} = -\frac{\eta g^2}{\Delta} (a a^+ |a\rangle \langle a| - a^+ a |b\rangle \langle b|) \]

\[ |c,0\rangle \rightarrow |c,0\rangle (|0,0\rangle \rightarrow |0,0\rangle) \]

\[ |b,0\rangle \rightarrow |b,0\rangle (|1,0\rangle \rightarrow |1,0\rangle) \]

\[ |c,1\rangle \rightarrow |c,1\rangle (|0,1\rangle \rightarrow |0,1\rangle) \]

\[ |b,1\rangle \rightarrow e^{i\phi} |b,1\rangle (|1,1\rangle \rightarrow e^{i\phi} |1,1\rangle) \]

\[ \phi = (g^2 / \Delta) t \]

• Quantum phase gate!!

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Cavity QED Based Quantum Phase Gate

\[ \left| a \right\rangle \quad \left| b \right\rangle \quad \left| c \right\rangle \]

\[ \omega_{ab} = \nu_2 \quad \omega_{bc} = \nu_1 + \Delta \]

Qubits: \( |0\rangle, |1\rangle \) in two modes of the cavity

Three-level atom in ground state \( |c\rangle \) passes through the cavity

\[ |0,0_2\rangle \rightarrow |0,0_2\rangle \]
\[ |0,1_2\rangle \rightarrow |0,1_2\rangle \quad \text{if} \quad \Delta \gg g_1 \]
\[ |1,0_2\rangle \rightarrow |1,0_2\rangle \]

\[ |0,0\rangle \rightarrow |0,0\rangle \]
\[ |0,1\rangle \rightarrow |0,1\rangle \]
\[ |1,0\rangle \rightarrow |1,0\rangle \]

\[ |1,1_2\rangle \rightarrow -|1,1_2\rangle \]

\[ \pi \text{-pulse phase shift} \]

Quantum dense coding

• One photon can carry one bit of information:

|0⟩ or |1⟩

Can one photon carry two bits of information?

• Initially Alice and Bob have entangled pair of photons

$$\frac{1}{\sqrt{2}} \left[ |0_A, 1_B⟩ + |1_A, 0_B⟩ \right]$$

Quantum dense coding

- Alice can now send two bits of information by sending her photon to Bob after making one of four operations:
  0,0 no change
  1,0 \( |0_A\rangle \rightarrow |1_A\rangle, \ |1_A\rangle \rightarrow |0_A\rangle \)
  0,1 \( |0_A\rangle \rightarrow |0_A\rangle, \ |0_A\rangle \rightarrow |1_A\rangle \)
  1,1 \( |0_A\rangle \rightarrow |1_A\rangle, \ |1_A\rangle \rightarrow |0_A\rangle \)

  Bob measures
  \[ \frac{1}{\sqrt{2}} [ |0_A,1_B\rangle + |1_A,0_B\rangle ] \]
  \[ \frac{1}{\sqrt{2}} [ |1_A,1_B\rangle + |0_A,0_B\rangle ] \]

- All operations are unitary

- Bob needs to make Bell basis measurement!!

Shor’s Algorithm for Factoring an Integer

• Problem:

Factoring a number $N$ into its prime factors

$$8463 = 3 \times 7 \times 13 \times 31$$

Input length $\log N$

• Conventional computers:

Factoring algorithm runs in $O\left(\exp\left(\frac{1}{3} \left(\ln N \right)^{\frac{1}{3}} \left(\ln \ln N \right)^{\frac{2}{3}}\right)\right)$ steps.
Factoring an integer

- $2^9 + 1$ (154 digit number) requires 700 workstations
- 200 digit number requires about 6 years
- 250 digit number requires about 80 years (RSA)
- 1000 digit number requires about $10^{12}$ years (Age of the universe ~$10^{10}$ years)
- Quantum Algorithm: (Shor, 1994)
  Requires about $O((\log N)^2)$ steps
  1000 digit number requires only few million steps
Why is this problem important?

Present day secure information transfer is based on public key system (RSA)

Encryption Key \((e,N)\)
Decryption Key \((d,N)\)

\(N = p \cdot q\) is a large number

\(\uparrow \uparrow\)

primes

(256 digits)

Code is broken if we can factorize \(N\)
Searching of an unsorted data base

Grover’s Algorithm

Search Problem
There is an unsorted data base containing $N$ items out of which just one item satisfies a given condition. That one item has to be retrieved.

On a classical computer, it would require an average of $N/2$ searches.

On a quantum computer, $O(\sqrt{N})$ steps are required. Also require $\log_2 N$ bits.
Quantum search without entanglement

M. O. Scully and M. S. Zubairy, Phys. Rev. A 64, 022304 (2001)

Classical search
\[ \left( \frac{pE}{1 \text{ Rabi freq}} \right) \tau = \pi \]

N pulses required—one for each atom

Energy required
\[ \left[ \left( \frac{1}{2} \varepsilon_0 E^2 \right) A \tau c \right] N = \mathcal{E} \]

‘Quantum’ search
\[ \left( \frac{p}{\eta \sqrt{N}} \right) \tau = \frac{\pi}{\sqrt{N}} \]

\[ \sqrt{N} \] pulses required

Energy required
\[ \left[ \left( \frac{1}{2} \varepsilon_0 E^2 \right) A \tau c \right] \sqrt{N} = \frac{\mathcal{E}}{\sqrt{N}} \]
Grover’s Algorithm \((N=4)\)

Two qubits: \(|0,0\rangle, |0,1\rangle, |1,0\rangle, |1,1\rangle\)

1. Prepare a quantum superposition of all states
   \[
   \frac{1}{2} \left[ |0,0\rangle + |0,1\rangle + |1,0\rangle + |1,1\rangle \right]
   \]

   \[
   \frac{1}{2} \left[ |0,0\rangle + |0,1\rangle + |1,0\rangle + |1,1\rangle \right]
   \]

2. Invert the desired state (say \(|0,1\rangle\))
   \[
   \frac{1}{2} \left[ |0,0\rangle - |0,1\rangle + |1,0\rangle + |1,1\rangle \right]
   \]

3. Inversion about mean
   \[
   \text{mean } x_0 = \frac{1}{4} \left[ \frac{1}{2} - \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right] = \frac{1}{4}
   \]

   \[
   \Rightarrow |0,1\rangle
   \]
Unitary Operations

Start with \[ |\Psi_0\rangle = |0^{\otimes k}\rangle \quad N = 2^k \]

(1) Walsh-Hadamard transformation \( W \) on all qubit
\[
|0\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle), \quad |1\rangle = \frac{1}{\sqrt{2}} (-|0\rangle + |1\rangle) \quad |\psi_1\rangle \equiv |s\rangle = \frac{1}{2} [ |0,0\rangle + |0,1\rangle + |1,0\rangle + |1,1\rangle ]
\]

(2) Invert the target state \( |t\rangle \)
For \( |t\rangle = |0,1\rangle \quad U_f = e^{i\pi |t\rangle\langle t|} = 1 - 2|t\rangle\langle t| \)
\[
U_f |\psi_1\rangle = \frac{1}{2} [ |0,0\rangle - |0,1\rangle + |1,0\rangle + |1,1\rangle ]
\]

(3) Inversion about mean
\[
U_s = e^{i\pi |s\rangle\langle s|} = 1 - 2|s\rangle\langle s| \quad U_s U_f |\Psi_1\rangle = |0,1\rangle
\]
Quantum interference

- Quantum eraser
- Quantum lithography with classical light
- Sub-wavelength resolution in microscopy
Erasing Knowledge!

As Thomas Young taught us two hundred years ago, photons interfere.

But now we know that:
Knowledge of path (1 or 2) is the reason why interference is lost. It's as if the photon knows it is being watched.

But now we discover that:
*Erasing the knowledge of photon path brings interference back.*

“No wonder Einstein was confused.”
Quantum interference

Quantum eraser

Time and the Quantum: Erasing the Past and Impacting the Future

Yakir Aharonov and M. Suhail Zubairy

The quantum eraser effect of Scully and Drühl dramatically underscores the difference between our classical conceptions of time and how quantum processes can unfold in time. Such eye-opening features of time in quantum mechanics have been labeled “the fallacy of delayed choice and quantum eraser” on the one hand and described as “one of the most intriguing effects in quantum mechanics” on the other. In the present paper, we discuss how the availability or erasure of information generated in the past can affect how we interpret data in the present.

The quantum eraser concept has been studied and extended in many different experiments and scenarios, for example, the entanglement quantum eraser, the kaon quantum eraser, and the use of quantum eraser entanglement to improve microscopic resolution.

The “classical” notion of time was summed up by Newton: “...absolute and mathematical time, of itself, and from its own nature, flows equally without relation to anything external.” In the present article, we go beyond our classical experience by presenting counterintuitive features of time as it evolves in certain experiments in quantum mechanics. To illustrate this point, an excellent example is the delayed-choice quantum eraser, proposed by Marlan O. Scully and Kai Drühl (1), which was described as an idea that “shook the physics community” when it was first published in 1992 (2). They analyzed a photon correlation experiment designed to probe the extent to which information accessible to an observer and its erasure affects measured results. The Scully-Drühl quantum eraser idea as it was described in Newsweek tells the story well (3), and Fig. 1 is an adaptation of their account of this fascinating effect.

In his book The Fabric of the Cosmos (4), Brian Greene sums up beautifully the counterintuitive outcome of the experimental realizations of the Scully-Drühl quantum eraser (p. 149):

These experiments are a magnificent eyes-on to our conventional notions of space and time. Something that takes place long after and far away from something else nevertheless is vital to our description of that something else. By any classical common sense reasoning, that’s, well, crazy. Of course, that’s the point—classical reasoning is the wrong kind of reckoning to use in a quantum universe... For a few days after I learned of these experiments, I remember feeling elated. I felt I’d been given a glimpse into a weird side of reality. Common experience—mundane, ordinary, day-to-day activities—suddenly seemed part of a classical charade, hiding the true nature of our quantum world. The world of the everyday suddenly seemed nothing but an inverted magic act, hiding in plain sight, into believing in the usual, familiar conceptions of space and time, while the astonishing truths of quantum reality lay carefully guarded by nature’s slights of hand.

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Erasing Knowledge!

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But now we know that:
Knowledge of path (1 or 2) is the reason why interference is lost. It's as if the photon knows it is being watched.

Fig. 1. Schematics for the Young's double-slit experiment. The which-path information wipes out the interference pattern. The interference pattern can be restored by erasing the which-path information.

No wonder Einstein was confused!
“Delayed choice” Quantum eraser

These experiments are a magnificent affront to our conventional notions of space and time. . . . . . . . . . . . . .For a few days after I learned of these experiments, I remember feeling elated. I felt I'd been given a glimpse into a veiled side of reality.
Young’s interference:

1st order correlation

\[
\langle I(r_1, r_1') \rangle = |E(1) + E(1')|^2
\]

\[
= 4 |E|^2 \cos^2 \left[ \frac{k(r_1 - r_1')}{2} \right]
\]

\[
= 4 |E|^2 \cos^2 \left[ \frac{k(d \sin \theta)}{2} \right]
\]

The first minimum occurs at

\[
k(d \sin \theta) / 2 = \pi / 2
\]

\[
\Rightarrow d = \frac{\lambda}{2 \sin \theta} \geq \frac{\lambda}{2}
\]

→ The diffraction limit.
Sub-wavelength microscopy

Two-photon correlation: Two Cascade Atoms

$$G^{(2)}_{\gamma \phi} = C \cos^2[(k_\gamma + k_\phi)(d \sin \theta)/2]$$

The first minimum occurs at $$k(d \sin \theta) = \pi / 2.$$ instead of $$k(d \sin \theta) / 2 = \pi / 2$$

$$\Rightarrow d = \frac{\lambda}{4 \sin \theta} \geq \frac{\lambda}{4}$$

Resolution improves by two fold!

Optical lithography

\[ I(x) \propto |e^{ikx} + e^{-ikx}|^2 = 2(1 + \cos(2kx)) \]

Rayleigh criterion:
Minimal resolvable feature size occurs at a spacing corresponding to the distance between an intensity maximum and an adjacent intensity minimum

For grazing incidence:
\[ 2kx_{\text{min}} = \pi \]
\[ x_{\text{min}} = \lambda / 2 \]
Classical two-photon exposure

Two-photon absorption probability:

\[ P_2 \propto I^2 \propto |e^{i k x} + e^{-i k x}|^4 \]

\[ = 3 + 4 \cos(2 k x) + \cos(4 k x) \]

IF \( \cos(2 k x) \) term can be neglected, then

\[ P_2 \propto 3 + \cos(4 k x) \]

\[ x_{\text{min}} = \lambda / 4 \]

Factor of 2 improvement !!

Two-Photon Quantum Lithography

Entangled state

\[
|\Psi\rangle = \frac{1}{\sqrt{2}} \left[ |0_A\rangle |2_B\rangle + |2_A\rangle |0_B\rangle \right]
\]

Deposition rate on a two-photon absorbing substrate

\[
P_2 \propto \langle \Psi | (c^\dagger + d^\dagger)^2 (c + d)^2 | \Psi \rangle = 1 + \cos(4kx)
\]

\[
c = (a - ib)e^{ikx} / \sqrt{2};
\]

\[
d = -(ia + b) / \sqrt{2}
\]

Difficulties:

• Entangled state generation for \(|N,0> + |0,N>\) is tedious.

• Two- or N-photon absorption probability is very low for low intensity beams.
Two-Photon Diffraction and Quantum Lithography

Milena D’Angelo, Maria V. Chekhova,* and Yanhua Shih

Department of Physics, University of Maryland, Baltimore County, Baltimore, Maryland 21250
(Received 2 February 2001; published 14 June 2001)

We report a proof-of-principle experimental demonstration of quantum lithography. Utilizing the entangled nature of a two-photon state, the experimental results have beaten the classical diffraction limit by a factor of 2. This is a quantum mechanical two-photon phenomenon but not a violation of the uncertainty principle.
Quantum Lithography with Classical Light

QUANTUM OPTICS

A New Way to Beat the Limits on Shrinking Transistors?

A new lithography scheme could sidestep a fundamental limit of classical optics and open the way to drawing ultraprecise patterns with simple lasers, a team of electrical engineers and physicists reports. If it works, the scheme would allow chipmakers to continue to shrink the transistors on microchips using standard technologies.

It seems quite cool, and it would be an advance in the field," says Jonathan Dowling, a mathematical physicist at Louisiana State University in Baton Rouge. Still, he says, researchers have a long way to go before they can put the plan into practice.

Chipmakers "write" the pattern of transistors and circuits on a microchip by shining laser light onto films called photore sist that lies atop a silicon wafer. According to classical optics, the light cannot create a pattern with details smaller than half its wavelength—the so-called diffraction limit. So to shrink transistors, chipmakers must use light of shorter wavelengths, such as ultraviolet light and soft x-rays. The problem is, ordinary lenses don't work at such short wavelengths.

Physicists know, in theory, they can beat the diffraction limit through quantum interference. They split a beam of light, send it through two slits, and combine the beam again. If the two slits are far apart, the pattern is smeared. But if the two slits are close together, the interference makes the pattern sharp. The problem is, ordinary lenses can't form images using light that's split in this way.

The combination beams can interfere with one another to make a pattern of bright and dark regions. The trick is to create a quantum connection called "entanglement" between a pair of photons traveling the two paths so that the two photons act as a single photon of twice the energy and half the wavelength. The interfering beams can then write details onto the photore sist that are smaller than the size of the diffraction limit. Entangling more photons produces smaller features still.

Such "quantum lithography" has yet to find its way into production lines, however, largely because it's hard to produce the entangled photons. Now, electrical engineer Philip Hemmer and physicist Shuji Nakamura of Texas A&M University in College Station and colleagues have concocted a scheme that they say can produce the same results with ordinary unentangled laser light.

Instead of splitting a beam, the researchers propose shining two "signal" lasers of slightly different wavelengths onto a surface coated with photore sist. They would also shine two "drive" lasers onto the same spot. In their scheme, the molecules of the photore sist can absorb a single photon of one of the drive beams. The phototransistors would effectively be patterned by beams of twice the energy and half the wavelength of each signal beam, yielding details half as small as the diffraction limit would allow, the researchers report in the 26 April Physical Review Letters.

The challenge will be to find just the right absorbing material for the phototransistors, says Dowling, one of the inventors of the entanglement approach. "I call this conservation of magic," he says. "Either you have to have a magic state of light, or you need a magic absorber." Yanhua Shih of the University of Maryland, Baltimore County, adds that the researchers have shown only that they can make a tight pattern of parallel lines. In principle, the new technique may not be able to make more elaborate patterns, Shih says. Others say the technique might produce the same result with ordinary unentangled laser light.

Such "quantum lithography" has yet to work. [Yan Hua Shih of the University of Maryland, Baltimore County, adds that the researchers have shown only that they can make a tight pattern of parallel lines. In principle, the new technique may not be able to make more elaborate patterns, Shih says. Others say the technique might produce the same result with ordinary unentangled laser light.

A new study that compares the immune responses of chimps and humans offers yet more compelling evidence that subtle differences in gene activity can result in big distinctions between the two species. The researchers, led by hematologist Ajit Varki of the University of California, San Diego (UCSD), suggest that their findings may explain why chimps and orangutans do not typically develop AIDS when infected with HIV, the disease that causes AIDS.

The team report in the 1 May Proceedings of the National Academy of Sciences. Varki and co-workers studied proteins called Siglecs that he has discovered in the 1990s. Many immune cells express Siglecs, which stand for sialic-acid-recognition Ig Superfamily lectins, and some of them appear to calm the immune response by preventing a process of immune cell expansion known as activation. Humans and apes share the same Siglec genes, but Varki's group explored whether they were turned on to the same degree in the lymphocytes taken from chimps, gorillas, and baboons. Using monoclonal antibodies to various Siglecs, the researchers found that although the 7 cells of people from many different geographic and ethnic backgrounds sported low levels of the Siglecs, this was not true of the chimpanzee. In the article, the authors write: "These results suggest that Siglec expression as a functional unit is a human-specific feature, and is unique to the species Homo sapiens."

5 May 2006
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Now consider a high frequency $\nu_0$. For any $N$ (= 1, 2, 3...) we can always find some suitable $n_N$ and $\omega_N$ to achieve the multiphoton wave vector $N\nu_0/c$. Two bunches of signal fields grazing from $+x$ and $-x$ directions will give the excitation rate

$$R^{(2nN-1)}(x,t) \propto \frac{d}{dt} \left| e^{i\frac{N\nu_0x}{c}} r_+^{(2nN-1)}(t) + e^{-i\frac{N\nu_0x}{c}} r_-^{(2nN-1)}(t) \right|^2. \quad (8)$$

For each $N$ we can use the above method to make an exposure. After multiple exposures we get fringes corresponding to $N = 1, 2, 3...N_{max}$. The final pattern is

$$P(x) = \int_0^t R(x,t')dt'$$

$$= \sum_{N=1}^{N_{max}} c_N \left| r^{(2nN-1)}(t_N) \left| e^{i\frac{N\nu_0x}{c}} + e^{-i\frac{N\nu_0x}{c}} e^{i\theta_N} \right|^2 \right|^2$$

$$= Q + \sum_{N=0}^{N_{max}} \left( a_N \cos \frac{2N\nu_0x}{c} + b_N \sin \frac{2N\nu_0x}{c} \right). \quad (9)$$

where $c_N$ is the ratio coefficient of $R^{(2nN-1)}$. Here $P(x)$ is a truncated Fourier series with an penalty deposition $Q$.

Q. Sun, P. R. Hemmer and M. S. Zubairy, (2007).
Two-dimensional patterns:

\[
F(x, y) = \begin{cases} 
  h & \text{if } -\frac{\pi}{2} < \frac{2\nu_0 x}{c}, \frac{2\nu_0 y}{c} < \frac{\pi}{2} \\
  0 & \text{elsewhere.}
\end{cases}
\]

Q. Sun, P. R. Hemmer and M. S. Zubairy, (2007).
Measurement of the separation between atoms beyond diffraction limit

Applications:

- Microscopy
- Nano-science
- Molecular level biology technology
- Lithography

Can the optical imaging resolution exceed Rayleigh diffraction limit?

Atom localization via resonance fluorescence


- Atom with a wavefunction $f(x)$ initially in level $|a>$
- Rabi frequency $\Omega$ through the standing wave is position dependent
- The spontaneously emitted photon carries information about position
- Atom with a wavefunction $f(x)$ initially in level $|a>$ passes through the standing wave.
- During the passage through standing wave it undergoes Rabi oscillations and eventually decays to level $|b>$

$$\Omega = \phi E / \hbar$$
Atom microscopy


Two Identical atoms: 1 and 2 with distance $\rho_{12}$

Establish a standing wave driving field along the direction of

$$\Omega_1 = \Omega \cdot \sin(k \cdot x_1) \quad \Omega_2 = \Omega \cdot \sin(k \cdot x_2)$$

$$x_1 = \frac{1}{k} \sin^{-1} \left( \frac{\Omega_1}{\Omega} \right) \quad x_2 = \frac{1}{k} \sin^{-1} \left( \frac{\Omega_2}{\Omega} \right)$$

distance $= x_1 - x_2$
Atomic energy part:

Dipole-dipole interaction:

Atom-field interaction:
Fluorescence spectrum properties

(i) Large distance: \[ \frac{\lambda}{2} \geq r_{12} \geq \frac{\lambda}{10} \quad \Omega_{12} \sim 0 \]

Exactly as the same as that in the simple scheme

\[ \Omega_1 = \Omega \cdot \sin(k \cdot x_1) \quad \Omega_2 = \Omega \cdot \sin(k \cdot x_2) \]
\[ x_1 = \frac{1}{k} \sin^{-1} \left( \frac{\Omega_1}{\Omega} \right) \quad x_2 = \frac{1}{k} \sin^{-1} \left( \frac{\Omega_2}{\Omega} \right) \]

distance = \[ x_1 - x_2 \]
Fluorescence Spectrum Properties

(ii) Very small distance: \( r_{12} \leq \frac{\lambda}{20} \), \( \Omega_{12} \gg \Omega_1, \Omega_2 \)

\[
\Omega_{12} = \frac{3}{2} \gamma \left\{ -\frac{\cos(kr_{ij})}{(kr_{ij})} + \frac{\sin(kr_{ij})}{(kr_{ij})^2} + \frac{\cos(kr_{ij})}{(kr_{ij})^3} \right\}
\]

We should have \( \Omega_{12} = \omega_0 \)

For \( \gamma \sim 10^7 \) Hz, \( \Omega_{12} \sim 10^{13} \) Hz, we estimate \( r_{12} \sim \lambda/550 \)
Fluorescence Spectrum Properties

(iii) Intermediate distance

\[ \frac{\lambda}{30} \leq r \leq \frac{\lambda}{10} \]

\( \Omega_{12} \approx \Omega_1, \Omega_2 \)

The spectrum turns out to be very complicated.

However, increase the intensity of the driving field

\( \Omega_1, \Omega_2 >> \Omega_{12} \)

Again,

\[ \Omega_{12} = \frac{3}{2} \gamma \left\{ -\frac{\cos(kr_{ij})}{kr_{ij}} + \frac{\sin(kr_{ij})}{(kr_{ij})^2} + \frac{\cos(kr_{ij})}{(kr_{ij})^3} \right\} \]

\[ \rightarrow r_{12} \]
Quantum thermodynamics
Reports

Extracting Work from a Single Heat Bath via Vanishing Quantum Coherence

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We present here a quantum Carnot engine in which the atoms in the heat bath are given a small bit of quantum coherence. The induced quantum coherence becomes vanishingly small in the high-temperature limit at which we operate and the heat bath is essentially thermal. However, the phase ̄φ, associated with the atomic coherence, provides a new control parameter that can be varied to increase the temperature of the radiation field and to extract work from a single heat bath. The deep physics behind the second law of thermodynamics is not violated; nevertheless, the quantum Carnot engine has certain features that are not possible in a classical engine.
Carnot Cycle:

- Isothermal expansion at $T=Th$, absorbing heat $Q_{in}$
- Adiabatic expansion until $T=Tc$
- Isothermal compression at $T=Tc$, rejecting $|Q_{out}|$
- Adiabatic compression until $T=Th$

Efficiency:

$$\eta_c = 1 - \frac{T_c}{T_h}$$
Photo-Carnot Engine

- Radiation pressure drives piston
- Single-mode radiation field at $T_{\text{rad}}$ by flowing atoms
- Atoms reheated by hohlraum at $T_{h}$

Steam engine with “photon gas”
without coherence

\[ \eta_c = 1 - \frac{T_c}{T_h} \]

with coherence

\[ \eta_\phi = \eta_c - \frac{T_c}{T_h} 3\bar{n}_\phi|\rho_{bc}| \cos \phi \]
Photo-Carnot Engine with Coherence

- 3-level atoms. Cavity frequency tuned to midway between ground-state doublet

- Atoms leave hohlraum at $T = T_h$

- Atoms acquire coherence
  $\rho_{bc} = |\rho_{bc}| e^{i\phi}$

- Coherent interaction with cavity radiation (like EIT)
  Temperature characterizing cavity radiation no longer $T_h$
Two level \[ \dot{n} = \alpha \rho_{aa}(T_h)(\bar{n} + 1) - \alpha \rho_{bb}(T_h)\bar{n} \] where \( \alpha = 2rg^2\tau^2 \)

At steady state and high T:
\[ \bar{n} = \frac{kT_h}{\eta\Omega} \]
\[ \eta_c = 1 - \frac{T_c}{T_h} \]

Three level with \( \rho_{bc} = |\rho_{bc}|e^{i\phi} \)
\[ \dot{n}_\phi = 2\alpha \rho_{aa}(\bar{n}_\phi + 1) - \alpha(\rho_{bb} + \rho_{cc} + 2|\rho_{bc}|\cos \phi)\bar{n}_\phi \]

At steady state, high T \( (\rho_{aa} = 1/3) \)
\[ \bar{n}_\phi = \frac{kT_h}{\eta\Omega}(1 - 3\bar{n}_\phi|\rho_{bc}|\cos \phi) \]

Cavity radiation temperature
\[ T_\phi = T_h(1 - 3\bar{n}_\phi|\rho_{bc}|\cos \phi) \]

leads to efficiency
\[ \eta_\phi = \eta_c - \frac{T_c}{T_h}3\bar{n}_\phi|\rho_{bc}|\cos \phi \]
The Bottom Line:

(Photos) Carnot: \( \eta_c = 1 - \frac{T_c}{T_h} \)

Photo-Carnot with coherence: \( \eta_\phi = \eta_c - \frac{T_c}{T_h} 3 \bar{n}_\phi \rho_{bc} |\cos \phi | \)

Hence for \( \phi = \pi \) \( \eta_\phi > \eta_c \) even if \( T_c = T_h \) !!

\[ W_{net} = \eta_\pi T_c \Delta S \approx \bar{n} \alpha_\mu \frac{\Delta \Omega}{\Omega} \eta \omega \]

But: \( W_{prep} > W_{net} \)
Implementation of Grover’s algorithm

\( |1\rangle \xrightarrow{U^1_{\pi/2}} |\pi/4\rangle \xrightarrow{\text{Quantum Phase Gate} Q_{\pi}} |\pi/4\rangle \xrightarrow{U^1_{\pi/2}} |\alpha\rangle\)

\( |1\rangle \xrightarrow{U^2_{\pi/4}} |\pi/8\rangle \xrightarrow{\text{Quantum Phase Gate} Q_{\pi}} |\pi/8\rangle \xrightarrow{U^2_{\pi/4}} |\beta\rangle\)

- Two qubits are represented by the modes of a cavity
- Single and two-bit gates are implemented by passing appropriate atoms through the cavity

\( W = \text{Walsh-Hadamard transformation} \)
\( C_{\alpha\beta} = \text{Oracle} \)
\( N = \text{Inversion about mean} \)

\[ w|0,0\rangle = \frac{1}{2} \left[ |0,0\rangle + |0,1\rangle + |1,0\rangle + |1,1\rangle \right] \]

\[ C_{\alpha\beta} \left[ w|0,1\rangle \right] = \frac{1}{2} \left[ |0,0\rangle - |0,1\rangle + |1,0\rangle + |1,1\rangle \right] \]

\[ N \left[ C_{\alpha\beta} \left[ w|0,1\rangle \right] \right] = |0,1\rangle \]
Discrete Quantum Fourier Transform

\[ |q\rangle \rightarrow \frac{1}{\sqrt{2^n}} \sum_{l=0}^{2^n-1} e^{2\pi i q p / 2^n} |p\rangle \]

\[ |q\rangle \equiv |q_n \ldots q_3 q_2 q_1 q_0 \rangle \]

q_i: qubits

\[ A_0 = A_1 = A_2 = \ldots = U_\frac{\pi}{4} \]

\[ B_{lm} = Q \frac{2\pi}{2^{m-l}} \]

Implementation of Grover’s Algorithm\textsuperscript{a}

\[ |1\rangle \xrightarrow{U^{1}_{\pi/4,-\pi/2}} |\text{Quantum phase gate}\rangle \xrightarrow{U^{2}_{\pi/4,-\pi/2}} |\text{Quantum phase gate}\rangle \xrightarrow{U^{2}_{\pi/2,0}} |\alpha\rangle \]

\[ |1\rangle \xrightarrow{U^{2}_{\pi/4,-\pi/2}} |Q_{\pi}\rangle \xrightarrow{U^{2}_{\pi/2,0}} |\beta\rangle \]

\textbf{W} \quad \textbf{C}_{\alpha,\beta}

(a) \quad |a_{2}\rangle \quad |a_{1}\rangle \quad |b\rangle

(b) \quad R_{1}R_{2} \quad R_{c}R_{3}R_{4}R_{5}R_{6} \quad R_{2}R_{7}R_{8}

\begin{array}{ccccccc}
4 & 3 & 2 & 1 & & & \\
\end{array}

\begin{array}{cccccccc}
\text{Atom 1} & & & & & & & \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\text{Atom 2} & & & & & & & \\
0 & 3 & 4 & 0 & 0 & 3 & 3 & 0 \\
\text{Atom 3} & & & \text{Atom 1} & & & & \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
\text{Atom 4} & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 \\
\end{array}

\text{Type A atom} \quad \text{Type B atom}

\begin{array}{ccccccc}
\text{(a)} & |a_{1}\rangle & |c_{2}\rangle & \quad |\nu_{1}\rangle & |\nu_{2}\rangle & |a_{2}\rangle & |b_{2}\rangle \\
\text{Type A atom} & |b_{1}\rangle & |c_{1}\rangle \\
\end{array}

\textbf{N}

\textbf{W} \quad \textbf{C}_{\alpha,\beta} \quad \textbf{N}

Grover’s Algorithm for $N = 2^n$

$K_m - m$th order quantum phase gate

$$K_m \left| \alpha_1, \alpha_2, \ldots, \alpha_m \right> = e^{i\pi \delta_{\alpha_1} \delta_{\alpha_2} \ldots \delta_{\alpha_m}} \left| \alpha_1, \alpha_2, \ldots, \alpha_m \right>$$

$\alpha_i = 0 \quad \text{or} \quad 1 (i = 1,2,\ldots,n)$

In terms of one-bit and two-bit gates:

Multiphoton quantum phase gate\textsuperscript{a}

For photons, effective Hamiltonian

\[ H_m = \eta \delta_m a_1^+ a_1 a_2^+ a_2 \ldots a_m^+ a_m \]

\[ Q^{(m)}_\eta |\alpha_1, \alpha_2, \ldots, \alpha_m\rangle = e^{i(\delta_m \tau)\alpha_1 \alpha_2 \ldots \alpha_m} |\alpha_1, \alpha_2, \ldots, \alpha_m\rangle \]

Consider a $\Lambda$ system

(Schmidt and A. Imamoglu, Opt. Lett. 21, 1936 (1996))

Effective Hamiltonian

\[ H_2 = \eta \delta_2 a_1^+ a_1 a_2^+ a_2 \]

with

\[ \delta_2 = \frac{g_2^2}{\Delta} \frac{g_1^2}{\Omega_1^2} \]

Homogeneously broadened medium: \( \delta_3 \tau \approx 3 \) rad

References

Entanglement


Quantum logic gates

References

Quantum search


Quantum Fourier transform

• Cavity QED implementation of the discrete quantum Fourier transform, M. O. Scully and M. S. Zubairy, Phys. Rev. A 65, 052324 (2002).

Quantum teleportation

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One-bit unitary gate

Possible initial states: $|0\rangle|b\rangle$ OR $|1\rangle|b\rangle$

- **First Step**
  
  $|0\rangle|b\rangle \rightarrow |0\rangle|b\rangle$  \hspace{0.5cm} $|1\rangle|b\rangle \rightarrow -i|0\rangle|a\rangle$

- **Second Step** (Interaction with classical field; $\theta = \Omega t$)
  
  $|a\rangle \rightarrow \cos \theta |a\rangle + ie^{-i\phi} \sin \theta |b\rangle$
  
  $|b\rangle \rightarrow ie^{i\phi} \sin \theta |a\rangle + \cos \theta |b\rangle$

- **Third Step**
  
  $|0\rangle|b\rangle \rightarrow |0\rangle|b\rangle$  \hspace{0.5cm} $|0\rangle|a\rangle \rightarrow -i|1\rangle|b\rangle$

**END RESULT:**

$|0\rangle|b\rangle \rightarrow (e^{i\phi} \sin \theta |1\rangle + \cos \theta |0\rangle)|b\rangle$  \hspace{0.5cm} $|1\rangle|b\rangle \rightarrow (- \cos \theta |1\rangle + e^{-i\phi} \sin \theta |0\rangle)|b\rangle$

---

Universality of quantum phase gate (QPG)

Consider
\[ Q_\pi = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix} \]

Recall
\[ U_\varphi = \begin{pmatrix}
\cos \frac{\varphi}{2} & \sin \frac{\varphi}{2} \\
-\sin \frac{\varphi}{2} & \cos \frac{\varphi}{2}
\end{pmatrix} \]

\[ U_{\pi/2} = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 1 \\
-1 & 1
\end{pmatrix} \]

\[ |0,0\rangle \rightarrow \frac{1}{\sqrt{2}}(|0,0\rangle - |0,1\rangle) \rightarrow \frac{1}{\sqrt{2}}(|0,0\rangle - |0,1\rangle) \rightarrow |0,0\rangle \]

\[ |0,1\rangle \rightarrow \frac{1}{\sqrt{2}}(|0,0\rangle + |0,1\rangle) \rightarrow \frac{1}{\sqrt{2}}(|0,0\rangle + |0,1\rangle) \rightarrow |0,1\rangle \]

\[ |1,0\rangle \rightarrow \frac{1}{\sqrt{2}}(|1,0\rangle - |1,1\rangle) \rightarrow \frac{1}{\sqrt{2}}(|1,0\rangle - |1,1\rangle) \rightarrow |1,1\rangle \]

\[ |1,1\rangle \rightarrow \frac{1}{\sqrt{2}}(|1,0\rangle + |1,1\rangle) \rightarrow \frac{1}{\sqrt{2}}(|1,0\rangle + |1,1\rangle) \rightarrow |1,0\rangle \]
Bell State Discriminator

- Bell basis (two qubits are the photons in two modes of the cavity):

\[ |\psi_1\rangle = \frac{1}{\sqrt{2}} \left( |0_1,1_2\rangle - |1_1,0_2\rangle \right) \quad |\psi_2\rangle = \frac{1}{\sqrt{2}} \left( |0_1,1_2\rangle + |1_1,0_2\rangle \right) \]

\[ |\psi_3\rangle = \frac{1}{\sqrt{2}} \left( |0_1,0_2\rangle - |1_1,1_2\rangle \right) \quad |\psi_4\rangle = \frac{1}{\sqrt{2}} \left( |0_1,0_2\rangle + |1_1,1_2\rangle \right) \]

- The operator

\[ U_{3\pi/4} Q_{\pi/4} U_{\pi/4}^2 U_{\pi/4}^1 \]

corresponds to the passage of appropriate atoms through the cavity.

- Leading to

\[ \psi_1 \rightarrow |0,0\rangle \quad \psi_2 \rightarrow |0,1\rangle \]

\[ \psi_3 \rightarrow |1,0\rangle \quad \psi_4 \rightarrow |1,1\rangle \]

Quantum Fourier transform

\[ |\psi(A, B)\rangle = \frac{1}{\sqrt{8}} \sum_{a=0}^{7} |a, f(a)\rangle \] (Entanglement)

\[ |a\rangle \rightarrow \frac{1}{\sqrt{8}} \sum_{a=0}^{7} e^{2\pi i \frac{ac}{8}} |c\rangle \] (QFT)

\[ \Rightarrow |\psi(A, B)\rangle = \frac{1}{8} \sum_{a=0}^{7} \sum_{c=0}^{7} e^{2\pi i \frac{ac}{8}} |c, f(a)\rangle \]

For \( f(a) = f(a + r) \) measurement of state of register A yields:

\[ |c_0\rangle = |0\rangle, |8 / r\rangle, |2 \times 8 / r\rangle \Lambda (r - 1)8 / r \]

\[ |\psi(A, B)\rangle = \frac{1}{2} \left[ |0, f(0)\rangle + |0, f(1)\rangle + |4, f(0)\rangle - |4, f(1)\rangle \right] \]

Let \( r = 2 \)

\[ f(0) = f(2) = f(4) = f(6) \]
\[ f(1) = f(3) = f(5) = f(7) \]

Outcome of measurement of register A is \(|0\rangle\) or \(|4\rangle\) with equal probability.
How to Factorize N?

- Select an integer \( x \) that is co-prime with \( N \)
- Find sequence formed by function \( f(a) = x^a \mod N \)
- Sequence has the form

\[
\begin{align*}
a & : 0, 1, 2, \Lambda, r-1, r, r+1, \Lambda \\
x^a & : 1, x, x^2, \Lambda, x^{r-1}, x^r, x^{r+1}, \Lambda \\
f(a) & : 1_4 4_2 4_3 \Lambda \Lambda \Lambda 1, x, \Lambda \Lambda \Lambda \Lambda
\end{align*}
\]

Sequence has periodic structure with period \( r \)
- Assume period \( r \) to be even
  \[
  1 = x^r \mod N
  \]
  or
  \[
  0 = (x^{r/2} - 1)(x^{r/2} + 1) \mod N
  \]

\( (x^{r/2} - 1) \) or \( (x^{r/2} + 1) \) must have common factors with \( N \)
- PROBLEM: Find \( r \) (period of function \( f(a) = x^a \mod N \) )
Example

\[ N = 91 \]
\[ x = 3 \]

\[ a \quad : \quad 0, \ 1, \ 2, \ 3, \ 4, \ 5, \ 6, \ 7, 8, \ldots \]

\[ 3^a \quad : \quad 1, \ 3, \ 9, \ 27, \ 81, \ 243, \ 729, \ 2187, \ldots \]

\[ f(a) \quad : \quad 1, \ 
\]

so

\[ 1 = 36 \ (\text{mod} \ 91) \]
\[ 0 = (3^3 - 1)(3^3 + 1) \ (\text{mod} \ 91) \]
\[ 0 = 26 \times 28 \ (\text{mod} \ 91) \]
\[ 0 = (2 \times 13) \times (2 \times 2 \times 7) \ (\text{mod} \ 91) \]

13, 7 are factors of 91
1) Prepare the following state in two registers

\[
\frac{1}{\sqrt{q}} \sum_{a=0}^{q-1} |a, f(a)\rangle \\
\text{with } f(a) = x^a \pmod{N}
\]

Highly entangled state!!!

2) Perform discrete ‘quantum’ Fourier transform on first register

\[
|a\rangle \rightarrow \frac{1}{\sqrt{q}} \sum_{a=0}^{q-1} e^{\frac{2\pi i ac}{q}} |c\rangle
\]

(reversible via inverse transform)

\[
\frac{1}{\sqrt{q}} \sum_{a=0}^{q-1} |a, f(a)\rangle \rightarrow \frac{1}{q} \sum_{a=0}^{q-1} \sum_{c=0}^{q-1} e^{\frac{2\pi i ac}{q}} |c, f(a)\rangle
\]
3) \[ \frac{1}{q} \sum_{a=0}^{q-1} \sum_{c=0}^{q-1} e^{2\pi i \frac{ac}{q}} |c, f(a)\rangle \]

- Retrieve the output from the quantum computer by measuring the state of all elements in the first register.

- If \( f(a) = f(a+r) \), the sum over \( a \) will yield constructive interference from coefficients \( e^{2\pi i \frac{ac}{q}} \).

- Only when \( \frac{c}{q} \) is a multiple of the reciprocal \( \frac{1}{r} \) period, all other values of \( \frac{c}{q} \) will produce destructive interference.

\[
\text{Proc}(c) = \begin{cases} 
0 & \frac{1}{r} \\
\frac{2}{r} & \frac{3}{r} \\
\frac{c}{q} 
\end{cases}
\]