Thermodynamics of McVittie Universe

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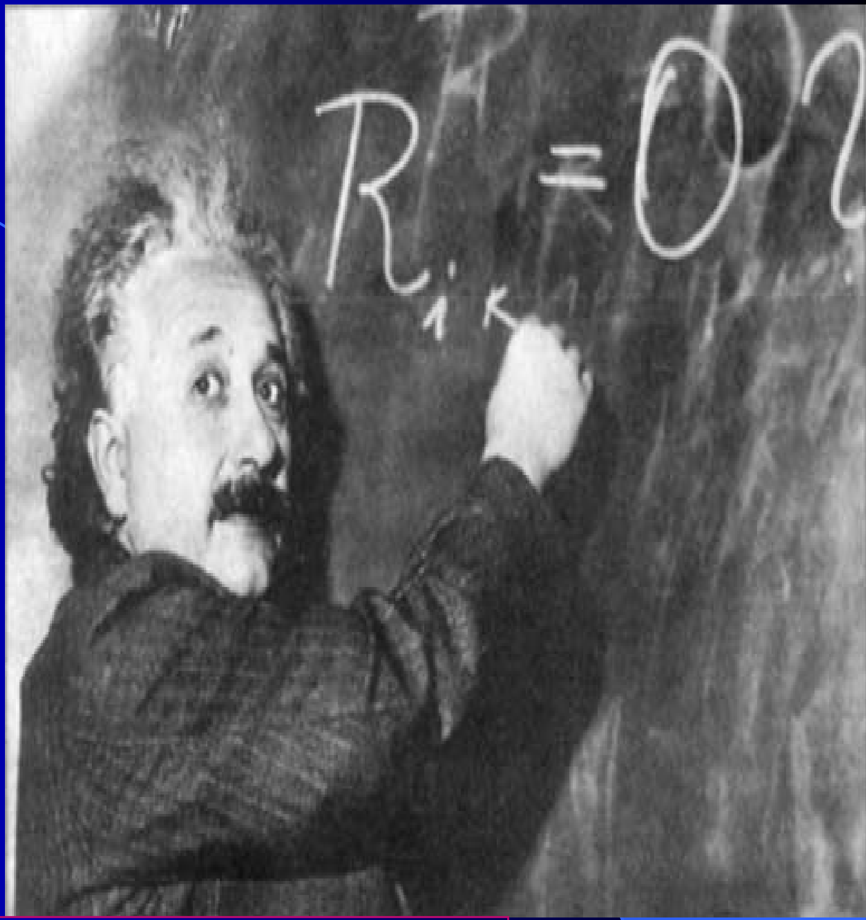
Einstein's Equations (1915):

Ricci Tensor

Gravitational Constant

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$



{Geometry matter (energy-momentum)}

C = 1

a) Brief Introduction to Black Hole Thermodynamics

1. Black Hole Solutions to the Einstein field equation
2. Cosmological solutions to the Einstein Field Equation

Black Hole (By Wheeler in 1967):
A spacetime region in which gravitational field is so strong that it precludes even light from escaping to infinity

Note: **Velocity required to leave the boundary of the black hole is equal to the speed of light (Classical Picture)**

Penrose (1967):

proposed that the black hole must contain non-zero entropy

Bekenstein, Bardeen, Carter, Hawking (1970-1974):

Jacob Bekenstein et al proposed that a black hole should have an entropy, and that it should be proportional to its horizon area.

Since black holes do not classically emit radiation, the thermodynamic viewpoint seemed simply an analogy.

Four Laws Black hole Thermodynamics ⟺ Four Laws of Thermodynamics

However, in 1974, Hawking applied quantum field theory to the curved spacetime around the event horizon and discovered that black holes emit Hawking radiation, a form of thermal radiation, which implied they had a positive temperature. He determined the constant of proportionality.

Schwarzschild Black Hole:

Units G = c =1

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

M = mass of the black Hole

Event Horizon → $r = 2M$

Surface Gravity → $\kappa = f'(r_+)/2$

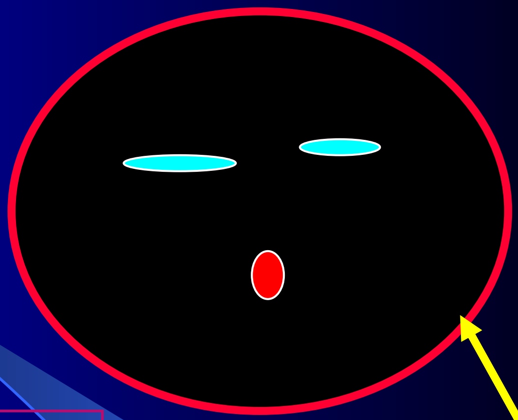
Temperature

$$T = \kappa / 2\pi$$

Surface gravity

Entropy

$$S = A/4G$$



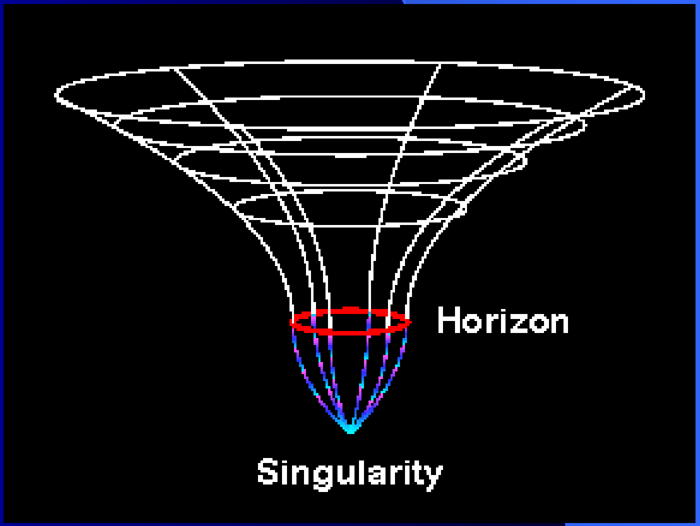
$A = 4\pi r^2$

horizon

Simple Example

Schwarzschild Black Hole (1916): Mass M

T dS = dM



More general:

Kerr - Newmann Black Holes (M, J, Q)

$$dM = T\,dS + \Omega\,dJ + \Phi\,dQ$$

Reference: Black Hole Physics, By Valeri P. Frolov and Igor D. Novikov

Einstein's Field Equation $|_{\text{(At horizon)}}$ ⟺ First Law of Thermodynamics

⇓

First by T. Jacobson

He found that it is indeed possible to derive the Einstein equations from the proportionality of entropy to the horizon area together with the fundamental relation $\delta Q = T\, dS$

T. Jacobson, Phys. Rev. Lett. 75 (1995) 1260

Thermodynamics of Spacetime: The Einstein Equation of State

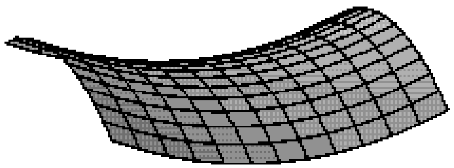
Horizon Thermodynamics: T. Padmanabhan, Class. Quant. Grav.(2002)[

c) From the First Law to the Friedmann Equations

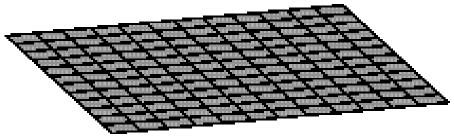
Friedman-Robertson-Walker Universe:

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right)$$

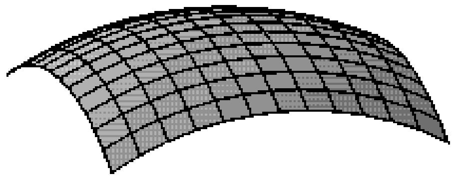
1) $k = -1$
open



2) $k = 0$
flat



3) $k = 1$
closed



Friedmann Equations:

$$H^2 + \frac{k}{a^2} = \frac{16\pi G}{n(n-1)}\rho.$$

$$\dot{H} - \frac{k}{a^2} = -\frac{8\pi G}{n-1}(\rho + p).$$

Where:

the Hubble parameter, $H \equiv \dot{a}/a$.

Our goal :

$$-dE = TdS.$$

$$H^2 + \frac{k}{a^2} = \frac{16\pi G}{n(n-1)}\rho.$$

$$\dot{H} - \frac{k}{a^2} = -\frac{8\pi G}{n-1}(\rho + p).$$

Some related works:

(1) A. Frolov and L. Kofman, JCAP 0305 (2003) 009

(2) Ulf H. Daniesson, PRD 71 (2005) 023516

(3) R. Bousso, PRD 71 (2005) 064024

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right)$$

$$ds^2 = h_{ab}dx^a dx^b + \tilde{r}^2 d\Omega_{n-1}^2,$$

where $\tilde{r} = a(t)r$ and $x^0 = t$, $x^1 = r$

Apparent Horizon in FRW Universe:

Apparent Horizon:

A marginally trapped surface with vanishing expansion

$$h^{ab}\partial_a \tilde{r} \partial_b \tilde{r} = 0.$$

$$\tilde{r}_A = \frac{1}{\sqrt{H^2 + k/a^2}},$$

Apply the first law to the apparent horizon:

$$-dE = TdS.$$

Make two ansatzes:

$$T = \frac{1}{2\pi R_A},$$

$$S = A/4G,$$

The only problem is to get *dE*

Suppose that the perfect fluid is the source, then

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu},$$

The energy-supply vector is:

$$\Psi_a = T_a{}^b \partial_b \tilde{r} + W \partial_a \tilde{r},$$

The work density is:

$$W = -\frac{1}{2} T^{ab} h_{ab},$$

S. A. Hayward, 1997,1998)

Then, the amount of energy crossing the apparent horizon within the time interval *dt*

$$-dE \equiv -A\Psi = A(\rho + p)H\tilde{r}_A dt,$$

$$-dE = TdS.$$

$$\dot{H} - \frac{k}{a^2} = -\frac{8\pi G}{n-1}(\rho + p).$$

By using the continuity equation:

$$\dot{\rho} + nH(\rho + p) = 0,$$

$$H^2 + \frac{k}{a^2} = \frac{16\pi G}{n(n-1)}\rho.$$

1. Cai and Kim, JHEP 0502 (2005) 050
2. Akbar and Cai, Phys. Rev. D75:084003,2007

McVittie Thermodynamics:

McVittie 1933:

McVittie Metric

$$ds^2 = -\frac{\left[1-\frac{M_0}{2a(t)r}\right]^2}{\left[1+\frac{M_0}{2a(t)r}\right]^2} dt^2 + a^2(t)\left[1+\frac{M_0}{2a(t)r}\right]^4 \cdot \left(dr^2 + r^2 d\Omega^2\right) ,$$

M_0 = Mass of the black hole

a (t) = scale factor of the universe

Notice That: $M_0 = 0$ → Flat FRW universe

a (t) = 1 → Isotropic form of Schwarzschild solution

Properties:

(i) The metric seems to represent a Schwarzschild black hole embedded in spatially flat FRW universe [D. J. Shaw et al Phys. Rev. D73 (2006) and others]

(ii) As $r \to \infty$, The McVittie metric reduces to a Robertson-Walker (RW) universe.

(iii) It is singular on the 2-sphere $r = M_0 / 2$. Also Pressure and Ricci scalar diverge at r = M0 / 2

McVittie can be rewritten as:

$$ds^2 = -\frac{A^2}{B^2}dt^2 + a(t)^2 B^4 dr^2 + R^2 d\Omega^2$$

Where

$$A = 1 - \frac{M_0}{2a(t)r} \qquad B = 1 + {M_0}/{2a(t)r} \qquad R = a(t)rB^2$$

Apparent Horizon:

$$h^{ab}\partial_a R \partial_b R = 0$$

$$h_{ab} = diag\left(-\frac{A^2}{B^2}, a(t)^2 B^4\right)$$

$$H^2 R_A^3 - R_A + 2M_0 = 0$$

Hubble parameter

Apparent Horizon

Cubic equation in apparent horizon radius

The different cases for the roots of the above equation are obtained by comparing it with
$x^3 + p x = q$,
having discriminant $D = (p/3)^3 + (q/2)^2$

$$D = -1/27 H^6 + M_0^2 / 4 H^4$$

$D > 0$ → One real root which is negative

$D = 0$ → all real roots and two are equal (one positive and one negative)

$D < 0$ → all roots are real and un-equal (two positive and one negative)

Entropy:

$$S = 4\pi R_A^2$$

Surface Gravity

$$\kappa = \frac{\dot{H}}{2H} + H^2 R_A - \frac{M_0}{R_A^2}$$

$$T = \kappa / 2\pi$$

Field Equation at apparent Horizon:

Misner - Sharp Energy

$$E = 4\pi/3 \; R_A^{\,2}\,\rho(t) + M_0$$

$$dS = 2\pi R_A \dot{R}\, dt$$

$$T\,dS = dE + P\,dW$$

Where $W = (1/2)(\rho - P)$ is the work density

Conclusion:

- **We worked out the explicit expressions for the apparent horizons of the McVittie universe.**
- **The temperatures associated with these horizon are obtained.**
- **It is also shown that these thermal quantities satisfy first law of thermodynamics.**

Thank You !