# Non-commutative field theories

Presented by:

Atta-ul-Latif

# Introduction

$$[x^i, x^j] = i\theta^{ij}$$

- Renormalizability at short distance
- Quantum Gravity
- String Theory
- UV-IR Mixing

# Steps to develop any Q.F.theory

**First get  Langrangian from action integrand**

**Get Hamiltonian**

**Go to quantum mechanics**

# Why, don't short-circuit

**1:- Hamiltonian  (difficult to make relativistic invariant )**

**2:- We have rule only for Lagrangian quadratic in velocity**

# Constraint on Hamiltonian formalism

**1:- primary constraint**

$$\Phi_m(q, p) = 0$$

**because** $p = p(\dot{q})$ **only**

**i.e. then q & p will linearly independent**

$$from \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_n}\right) = \left(\frac{\partial L}{\partial q_n}\right)$$

**=====>** **Important to note: lagrangian is not completely arbitrary**

**Important to note**

**1=0 for simple case L=q**
**i.e. not consistence**

# So Hamiltonian " H "

$$H = p_n \dot{q}_n - L$$

*become*

$$H^* = H + c_m f_m \quad \text{because } \Phi_m = 0$$

**=> Hamiltonian is not uniquely determine**

$$\{\mathscr{A}, \mathscr{B}\}_{DB} = \{\mathscr{A}, \mathscr{B}\} - \{\mathscr{A}, X_s\}\, X_{ss'}\, \{X_{s'}, \mathscr{B}\}$$

$$X_{ss'} = \Delta^{-1} = \begin{pmatrix} \{T^\alpha, T^\beta\} & \{T^\alpha, G_\beta\} \\ \{G_\alpha, T^\beta\} & \{G_\alpha, G_\beta\} \end{pmatrix}^{-1}$$

# Usual field theories

$$\left.\begin{aligned}\{q^\alpha, q^\beta\}_{DB} &= 0\,, \\ \{q^\alpha, p_\beta\}_{DB} &= \delta^\alpha_\beta\,, \\ \{p_\alpha, p_\beta\}_{DB} &= 0\,,\end{aligned}\right\}$$

$$\left.\begin{aligned}\{q^\alpha, v_\beta\}_{DB} &= \delta^\alpha_\beta\,, \\ \{q^\alpha, \pi^\beta\}_{DB} &= 0\,.\end{aligned}\right\}$$

# Non-commutative field theories

$$\text{a}), \quad \{q^{\alpha}, q^{\beta}\}_{DB} = -2\,\theta^{\alpha\beta}$$

$$\text{b}), \quad \{q^{\alpha}, p_{\beta}\}_{DB} = \delta^{\alpha}_{\beta}$$

$$\text{c}), \quad \{p_{\alpha}, p_{\beta}\}_{DB} = 0$$

$$\text{d}), \quad \{q^{\alpha}, v_{\beta}\}_{DB} = \delta^{\alpha}_{\beta}$$

$$\text{e}), \quad \{q^{\alpha}, \pi^{\beta}\}_{DB} = -\theta^{\alpha\beta}$$

# Correspondence transformation

- **By transforming the physical variables as**

$$q^\alpha \longrightarrow \mathbb{Q}^\alpha \equiv q^\alpha + \theta^{\alpha\beta} p_\beta$$

$$p^\alpha \longrightarrow \mathbb{P}_\alpha \equiv p_\alpha$$

- **By replacing all usual product with star *-product**

$$\psi(q) \star \phi(q) \equiv e^{\frac{i\theta^{\alpha\beta}}{2} \frac{\partial}{\partial \xi^\alpha} \frac{\partial}{\partial \zeta^\beta}} \psi(q+\xi)\,\phi(q+\zeta)\Big|_{\xi=\zeta=0}$$

# Types of non-commutative field theories

1. General space-time non-commutative theories, i.e., $\theta_{0i} \neq 0$

2. Particular space non-commutative theories, i.e., $\theta_{0i} = 0$

# CPT Symmetry

- Parity (P)

1. **Parity transformation are same as usual QED for space-space non-commutativety**

**but**

2. **For general space-time non-commutativety, Parity (P) is broken**

**Because we have to replace** $q_{0i}$ **by** $-q_{0i}$ **along usual transformation**

**but there should be no change in** $q_{ij}$

- Charge Conjugation

**For both space-space and space-time non-commutivity**

$$q \rightarrow -q$$

**along with usual transformation**

- Time reversal (T)

1. **For space-space non-commutativety along usual Time reversal transformation, we have to replace**

$$q \rightarrow -q$$

2. **For general space-time non-commutativety, Time reversal (T) is broken**

**Because we have to** $q_{ij} \rightarrow -q_{ij}$

**But there should no change in** $q_{0i}$

# CP and CT Symmetry

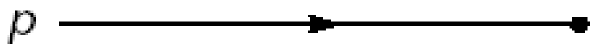
- For Space-Space non-commutativity

  1. **CP violate**
  2. **CT is invariant**

- For Space-Time non-commutativity

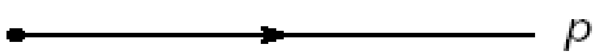
**both CP and CT is broken as C, T and P are all separately broken**

# Feynman rules for NCQED

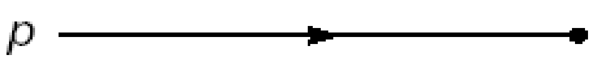



For each incoming electron : $u_r(\mathbf{p})$


For each outgoing electron : $\bar{u}_r(\mathbf{p})$




For each incoming positron : $\bar{v}_r(\mathbf{p})$

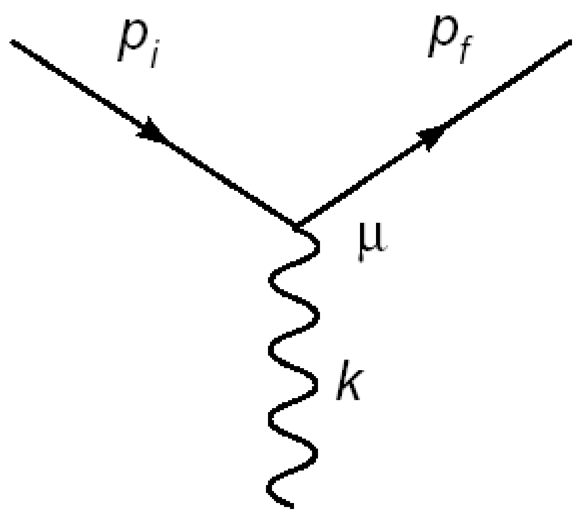



For each outgoing positron : $v_r(\mathbf{p})$


For each incoming photon : $\varepsilon_{r\mu}(\mathbf{k})$

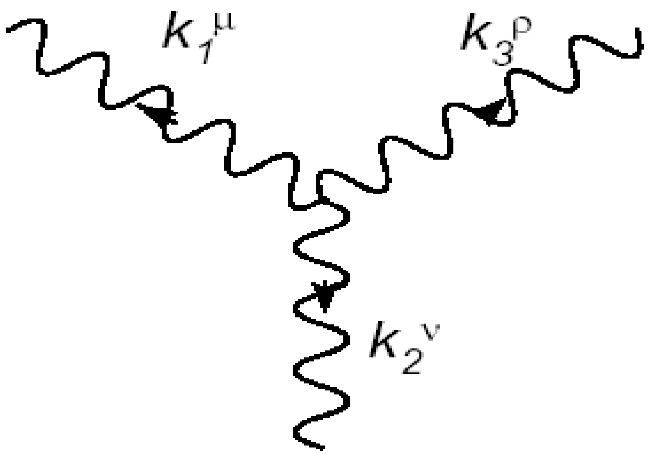

For each outgoing photon : $\varepsilon^{*}_{r\mu}(\mathbf{k})$

1. Fermion-fermion-photon vertex:




$$= ie\gamma^{\mu} e^{\frac{i}{2} p_i \times p_f}$$

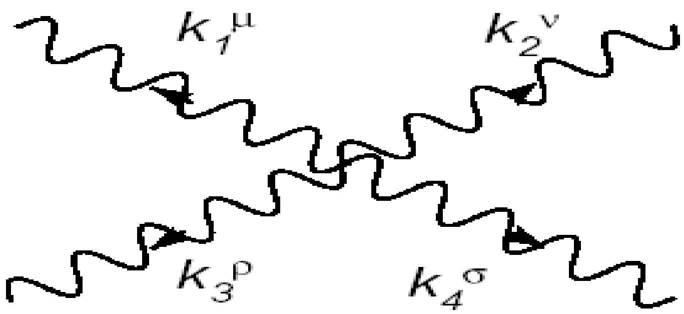
2. Photon-photon-photon vertex:




$$= -2e \sin\left(\frac{1}{2} k_1 \times k_2\right) C^{\rho\mu\nu}(k_1, k_2, k_3)$$

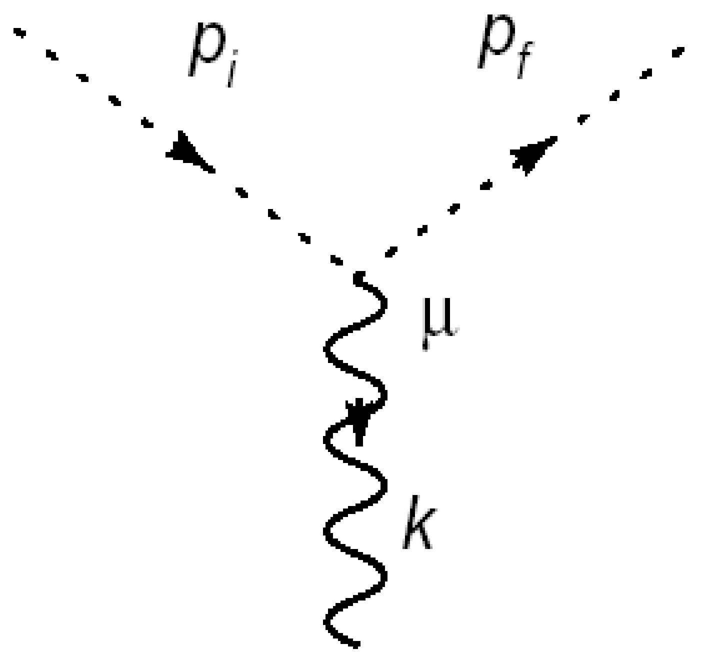
$$= -2e \sin\left(\frac{1}{2} k_1 \times k_2\right) \times \Big[(k_1 - k_2)^\rho g^{\mu\nu}$$
$$+ (k_2 - k_3)^\mu g^{\nu\rho} + (k_3 - k_1)^\nu g^{\rho\mu}\Big]$$

3. Photon-photon-photon-photon vertex:




$$
\begin{aligned}
& -4ie^2\Big[\,(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\
& \times \sin\left(\frac{k_1 \times k_2}{2}\right)\sin\left(\frac{k_3 \times k_4}{2}\right) \\
& + (g^{\mu\sigma}g^{\nu\rho} - g^{\mu\nu}g^{\rho\sigma}) \\
& \times \sin\left(\frac{k_3 \times k_1}{2}\right)\sin\left(\frac{k_2 \times k_4}{2}\right) \\
& + (g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}) \\
& \times \sin\left(\frac{k_1 \times k_4}{2}\right)\sin\left(\frac{k_2 \times k_3}{2}\right)\Big].
\end{aligned}
$$

4. Ghost-ghost-photon vertex




$$= \quad 2iep_f^{\mu} \sin(\frac{1}{2} p_i \times p_f)$$

# Radioactive correction

- One loop correction for electron self-energy

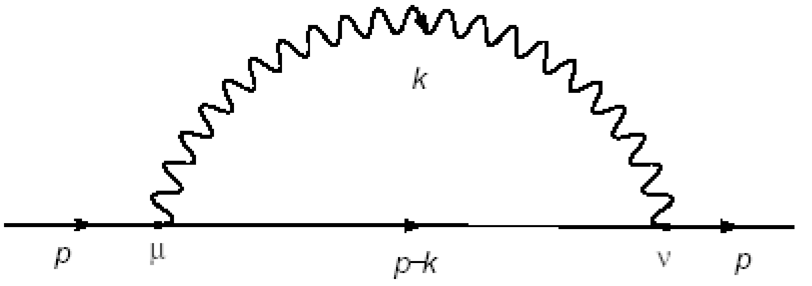



Figure 4.1: Electron self energy diagram

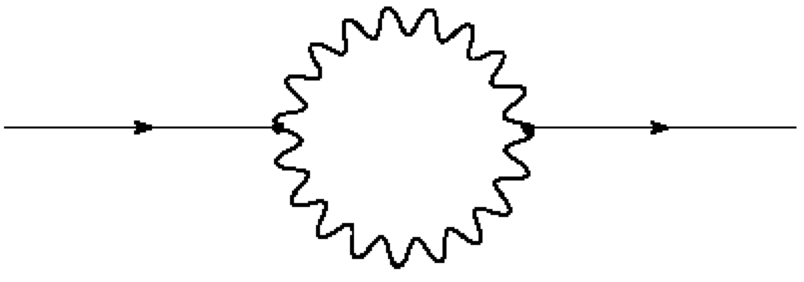


Figure 4.2: The diagram not possible

# Photon self-energy

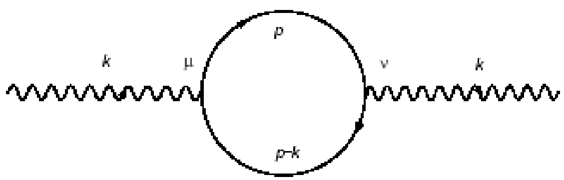



Figure 4.3: Photon self energy due to fermion loop correction

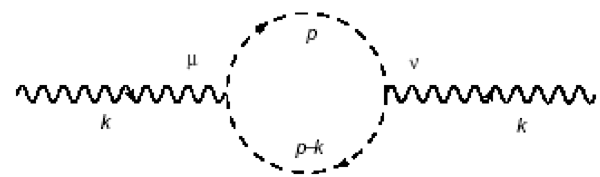



Figure 4.4: Photon self energy due to ghost-loop correction

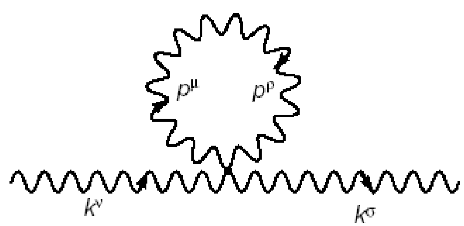



Figure 4.5: Photon self energy due to photon loop correction with one vertex

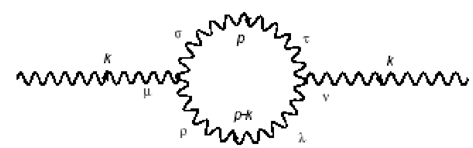


Figure 4.7: Photon self energy due to photon loop correction with two vertex

# Electron-Photon vertex correction

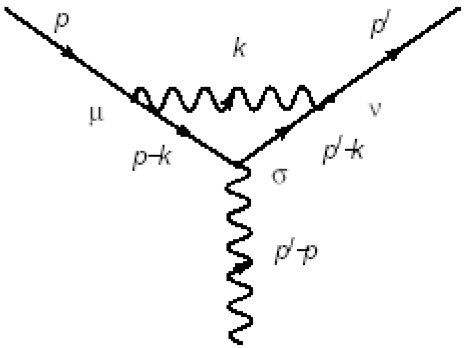



Figure 4.8: Vertex correction for QED like diagram

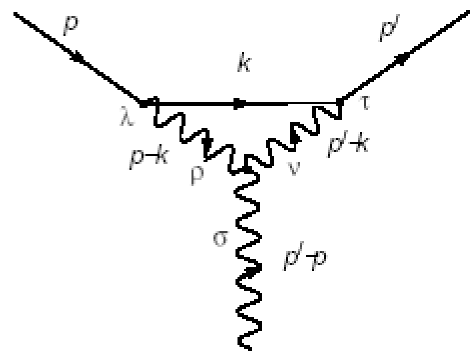



Figure 4.9: Vertex correction for QCD like diagram

# UV/IR Mixing

$$\Gamma^{\sigma(UV-ren)} = \frac{-\alpha e^{\frac{i}{2}p\times p'}}{\pi}\int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 \delta(1-\sum\alpha_i)\times$$
$$\times\Bigg(\frac{A^\sigma e^{-i(\alpha_2+\alpha_3)p.\tilde{q}}}{\alpha_1 m_\gamma^2+(\alpha_2+\alpha_3)^2 m^2-\alpha_2\alpha_3 q^2}+$$
$$+\frac{\bar{A}^\sigma(1-e^{i(\alpha_2+\alpha_3)p.\tilde{q}}e^{-ip\times p'})}{m^2(\alpha_1-\alpha_2-\alpha_3)+m_\gamma^2(\alpha_2+\alpha_3)+m^2(\alpha_2+\alpha_3)^2-\alpha_2\alpha_3 q^2}-$$
$$-2\gamma_{Euler}(B^\sigma e^{-i(\alpha_2+\alpha_3)p.\tilde{q}}+\bar{B}^\sigma-\tilde{B}^\sigma e^{i(\alpha_2+\alpha_3)p.\tilde{q}}e^{-ip\times p'})+$$
$$+\Lambda_{eff}^2(C^\sigma e^{-i(\alpha_2+\alpha_3)p.\tilde{q}}-\tilde{C}^\sigma e^{i(\alpha_2+\alpha_3)p.\tilde{q}}e^{-ip\times p'})\Bigg). \qquad (4.2.66)$$

$$C^\sigma = \frac{-\gamma^\sigma \tilde{q}.\tilde{q}}{8} + \frac{\tilde{q}^\sigma \gamma.\tilde{q}}{4}$$

**and**

$$\tilde{C}^\sigma = \frac{\gamma.\tilde{q}\tilde{q}^\sigma}{4} + \frac{\gamma^\sigma \tilde{q}.\tilde{q}}{8}$$

$$\Lambda^2_{eff} = \frac{1}{\Lambda^{-2} - \frac{\tilde{q}.\tilde{q}}{4}} \, .$$

**For UV limit** $$\frac{1}{\Lambda^2} \ll \tilde{q}.\tilde{q} \Rightarrow \Lambda_{eff} \sim \frac{1}{\tilde{q}.\tilde{q}}$$ **(Term is finite)**

**For IR limit** $$\frac{1}{\Lambda^2} \gg \tilde{q}.\tilde{q} \Rightarrow \Lambda_{eff} \sim \Lambda^2$$

**(Seems to lead IR divergence)**

**But** $\Lambda^2 \tilde{q}^2 \ll 1$ **in IR limit.**

**Therefore** $C^s$ **terms are irrelevant.** **(Term is again finite)**

# Photon deflection by coulomb field

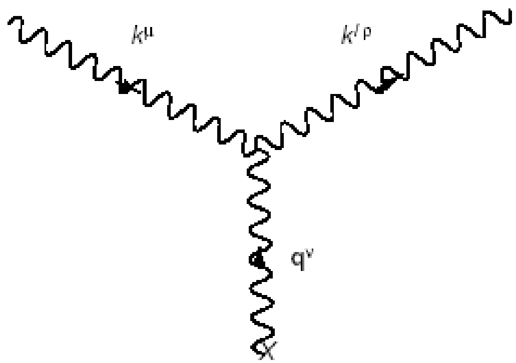



Figure 4.10: Photon deflection by coulomb Field

**For** $\theta_{01} = \theta_{03} = 0$ **and** $\theta_{13} = 1$

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2E^2}{4\Lambda^4}\frac{(1+2\sin^2(\phi/2))(1-\sin^2(\phi/2))}{\sin^2(\phi/2)}$$

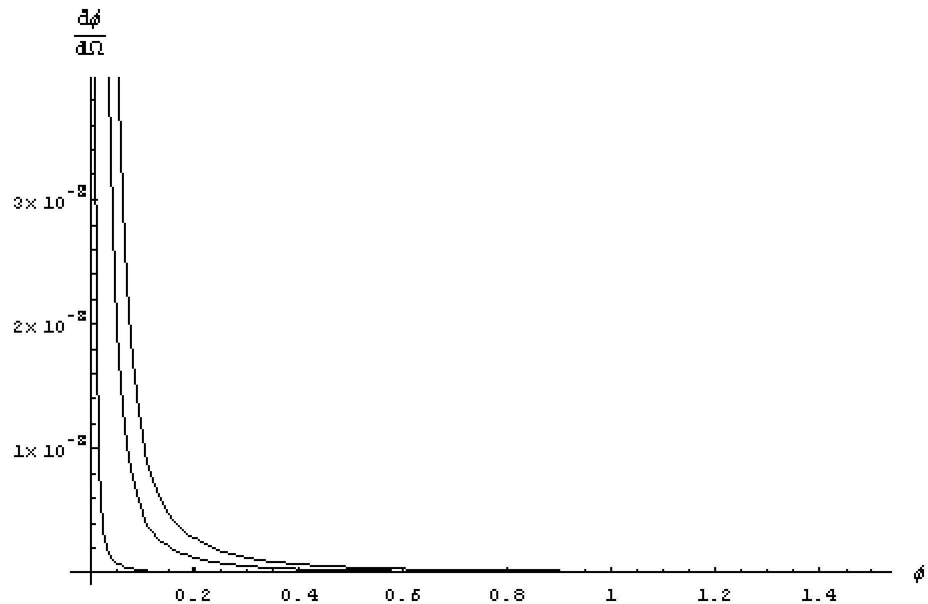



Figure 4.11: The behavior of differential cross section with the scattering angle $\phi$ for scattered photon due to coulomb field with $Z = 1, \alpha = 1/128$ & $\Lambda = 1500$ GeV with different values of E = 400, 2000 and 3000.

# Interpretation to radioactive correction

- **Vertex correction**

Correction to the electron response to a given applied field

- **Loop correction**

Correction to the electro magnetic field itself

# Interpretation of different terms of non-commutative vertex

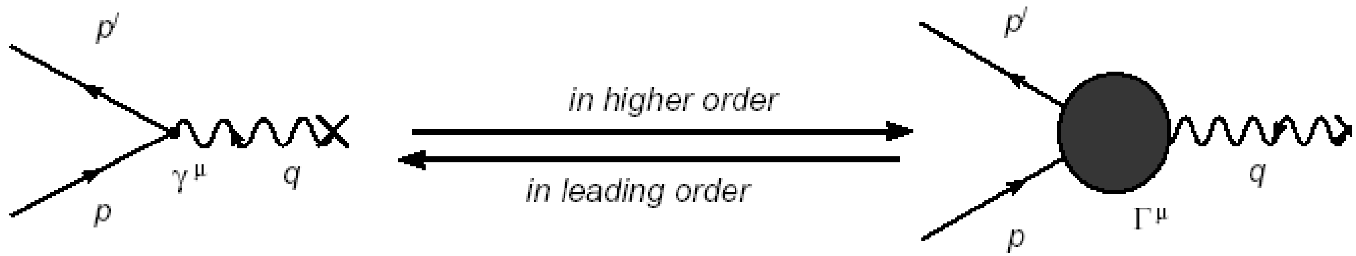



Figure 5.1: Vertex correction

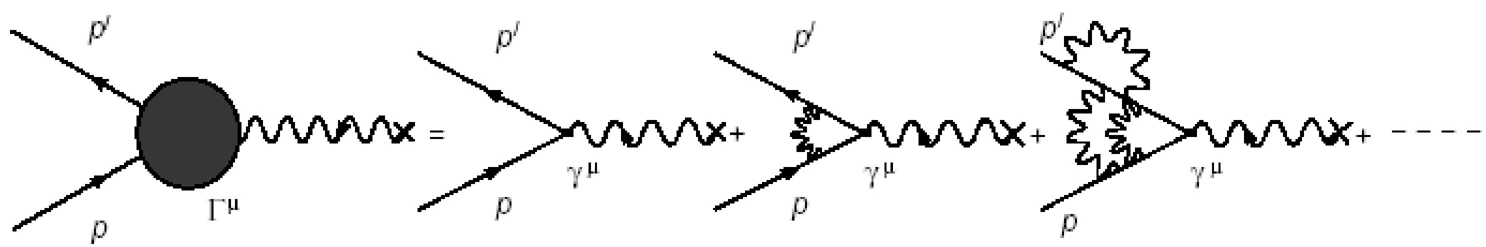



Figure 5.2: Vertex correction

$$\mathcal{M} = ie\bar{u}(p')\gamma^\mu u(p)\tilde{A}_\mu(q) \xrightarrow{\text{In higher order}} \mathcal{M} = ie\bar{u}(p')\Gamma^\mu u(p)\tilde{A}_\mu(q)$$

**Where for usual QED**

**List of vectors :** $p^\mu,\ p'^\mu,\ \gamma^\mu.$ **List of scalars :** $q^2,\ m,\ e$ .

**Therefore we have**

$$\Gamma^\mu = F_1(q^2)\gamma^\mu + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2)$$

**At tree level** $F_1(q^2) = 1 \text{ and } F_2(q^2) = 0.$

**In coulomb gauge** $\tilde{A}_u(q) = (\tilde{A}_0, 0)$

$$\mathcal{M} = ie\tilde{\phi}(q)F_1(0)$$

$$V(x) = ie\tilde{\phi}(q)F_1(0)$$

$F_1(0)$ represents the electric charge of the electron

**For static vector potential** $\tilde{A}_u(q) = (0, \tilde{A}_i)$

$$\mathcal{M} = ie\, 2m\, \xi^\dagger \frac{-i}{2m} \epsilon^{ijk} q_j \sigma_k \Big[ F_1(0) + F_2(0) \Big] \xi \tilde{A}_i(q)$$

$$\tilde{B}^k(q) = -i\epsilon^{ijk} q_j \tilde{A}_i(q)$$

$$\mathcal{M} = ie\, \xi^\dagger \sigma_k \left[ F_1(0) + F_2(0) \right] \xi \tilde{B}^k(q)$$

$$V(x) = -\langle \mu \rangle . B(x)$$

$$\langle \mu \rangle = \left[ F_1(0) + F_2(0) \right] \left( \frac{e}{2m} \right) \xi^\dagger \sigma \xi$$

# Vertex in non-commutative field theory

$$\Gamma^{\mu} \xrightarrow{\text{at tree level}} e^{\frac{i}{2}p.\tilde{q}'}\gamma^{\mu}$$

**List of scalars :** $m,\ e,\ q^2,\ \underbrace{\tilde{q}.\tilde{q},\ \gamma.\tilde{q},\ p.\tilde{q}}$ **List of vectors :** $\gamma^{\mu},\ (p'+p)^{\mu},\ q^{\mu},\ \tilde{q}^{\mu}$

$$\Gamma^{\mu} = A\gamma^{\mu} + B(p'+p)^{\mu} + Cq^{\mu} + D\tilde{q}^{\mu}$$

$$\Gamma^{\mu} = F_1(\ q^2,\ p.\tilde{q},\ \gamma.\tilde{q})\gamma^{\mu} + \frac{i\sigma^{\mu\nu}q_{\nu}}{2m}F_2(\ q^2,\ p.\tilde{q},\ \gamma.\tilde{q}) + D(\ q^2,\ p.\tilde{q},\ \gamma.\tilde{q})\tilde{q}^{\mu}$$

$$\Gamma^{\mu} = F_{1(a)}\gamma^{\mu} + (F_{1(b)}p.\tilde{q})\gamma^{\mu} + \frac{i\sigma^{\mu\nu}q_{\nu}}{2m}\left[F_{2(a)} + (F_{2(c)}\gamma.\tilde{q})\right] + D_{(a)}\tilde{q}^{\mu}$$

- **The term containing $F_{1(a)}\gamma^{\mu}$**

$$F_{1(a)}(q^2) = \frac{-\alpha e^{\frac{i}{2}p\times p'}}{\pi}\int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 \delta(1-\sum\alpha_i)\times\left(1-e^{i(\alpha_2+\alpha_3)p.\tilde{q}}e^{-ip\times p'}\right)$$

$$\times\left\{\left[\frac{(2p'.p-(\alpha_2+\alpha_3)(p'+p)^2)+m^2(\alpha_2+\alpha_3)^2-\alpha_2\alpha_3 q^2)}{2(\alpha_1 m_\gamma^2+(\alpha_2+\alpha_3)^2 m^2-\alpha_2\alpha_3 q^2)}+\gamma_{Euller}\right]e^{-i(\alpha_2+\alpha_3)p.\tilde{q}}\right.$$

$$\left.+\left[\frac{((\alpha_2+\alpha_3)(p'+p)^2-3m^2-m^2(\alpha_2+\alpha_3)^2+\alpha_2\alpha_3 q^2)}{m^2(\alpha_1-\alpha_2-\alpha_3)+m_\gamma^2(\alpha_2+\alpha_3)+m^2(\alpha_2+\alpha_3)^2-\alpha_2\alpha_3 q^2}-\frac{3\gamma_{Euller}}{2}\right]\right\},$$

**The term containing** $(F_{1(b)} p.\tilde{q})\gamma^{\mu}$

$$F_{1(b)} = \frac{-\alpha e^{\frac{i}{2} p\times p'}}{\pi} (i\gamma_{Euller}) \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 \delta(1 - \sum \alpha_i)$$

$$\left(1 - e^{-ip\times p'}\right)(2 - \alpha_2 - \alpha_3) e^{-i(\alpha_2+\alpha_3)p.\tilde{q}}$$

**The term containing $\frac{i\sigma^{\mu\nu}q_\nu}{2m}F_{2(a)}$**

$$
\begin{aligned}
F_{2(a)} &= \frac{-\alpha e^{\frac{i}{2}p\times p'}}{\pi}\int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 \delta(1-\sum\alpha_i)\times \\
&\times\left\{\frac{m\alpha_1(\alpha_2+\alpha_3)e^{i(\alpha_2+\alpha_3)p.\tilde{q}}}{\alpha_1 m_\gamma^2+(\alpha_2+\alpha_3)^2m^2-\alpha_2\alpha_3q^2}+\right. \\
&\left.+\frac{m\alpha_1(\alpha_2+\alpha_3)\left(1-e^{i(\alpha_2+\alpha_3)p.\tilde{q}}e^{-ip\times p'}\right)}{m^2(\alpha_1-\alpha_2-\alpha_3)+m_\gamma^2(\alpha_2+\alpha_3)+m^2(\alpha_2+\alpha_3)^2-\alpha_2\alpha_3q^2}\right\}
\end{aligned}
$$

The term containing $\frac{i\sigma^{\mu\nu}q_\nu}{2m}(F_{2(c)}\gamma.\tilde{q})$

$$F_{2(c)} = \frac{-\alpha e^{\frac{i}{2}p\times p'}}{\pi}\frac{i\gamma_{Euller}}{2}\int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 \delta(1-\sum\alpha_i)\times$$
$$\left((2-\alpha_2-\alpha_3)e^{-i(\alpha_2+\alpha_3)p.\tilde{q}} + (1+\alpha_2+\alpha_3)e^{i(\alpha_2+\alpha_3)p.\tilde{q}}e^{-ip\times p'}\right)$$

**This term gives electric dipole moment of the form**

$$\langle P\rangle_i = 2iF_{2(c)}\tilde{p}_i$$

Néda Sadooghi and M.Soroush, "Noncommutative Dipole QED", hep-th/0206009.

**The term containing $D_{(a)}\tilde{q}^{\mu}$**

$$D_{(a)} = \frac{-\alpha e^{\frac{i}{2}p\times p'}}{\pi} i\, m\, \gamma_{Euller} \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 \delta(1-\sum\alpha_i) \times$$
$$\times \left( (1+\alpha_2+\alpha_3) e^{-i(\alpha_2+\alpha_3)p.\tilde{q}} - (2-\alpha_2-\alpha_3) e^{i(\alpha_2+\alpha_3)p.\tilde{q}} e^{-ip\times p'} \right)$$

**This term gives magnetic moment of the form**

$$\langle\mu\rangle = \frac{D_{(a)}}{i}\vec{\theta}, \quad \text{with } \theta_i \equiv \epsilon_{ijk}\theta_{jk}$$

Therefore it is straight forward to see that the magnetic moment due to this term does not depend upon the spin. It is pure non-commutative aspect.

J.Gamboa, M.Loewe, F.Méndez and J.C.Rojas, "The Landau problem and noncommutative quantum mechanics", hep-th/0104224.

H. O. Girotti, "Noncommutative quantum mechanics", Am. J. Phys. **72**(5), May 2004.

We have observed that magnetic moment in non-commutative electron photon vertex has two parts

1. One spin dependent part,

2. Other is spin independent part, which is proportional to $\theta^{\mu\nu}$ and have lost contribution in the limit $\theta \to 0$.

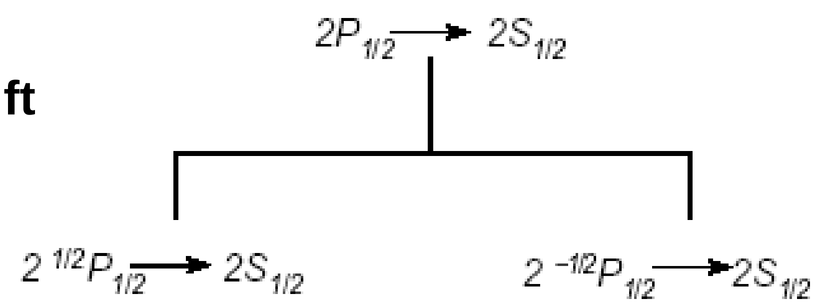
# Loop correction (The Lamb Shift in NCQED)

- Hydrogen atom in NCQM (at tree level)

$$\Delta E = -\frac{m_e c^2}{4}(Z\alpha)^4 \frac{\theta}{\lambda_e^2} j_z \left(1 \mp \frac{1}{2l+1}\right)\left(\frac{1}{n^3 l(l+1)(l+1/2)}\right)\delta_{ll'}\delta_{j_z j_z'}$$

**For** $n^{\,j_z} l_j$

**Polarized lamb-shift**



# One loop correction

- Vertex correction (two different types of diagram)

$$V_{Vertex} = -\frac{Ze^2}{4\pi}\alpha\gamma_{Euler}\left(3-\frac{2}{3}\right)\frac{\vec{L}.\vec{\theta}}{\hbar r^3}$$

- Loop correction (four different types of diagram)

$$V_{Loop}(r) = -Ze^2\alpha\frac{10}{3\hbar}\int d^3q\frac{1}{\vec{q}^{\,2}}e^{-i\vec{q}.\vec{r}/\hbar}\left[\ln\left(\frac{\vec{q}^{\,2}\tilde{q}^{\,2}}{\hbar^4}\right)+\frac{2}{25}\frac{\vec{q}^{\,2}}{m^2}\right]$$

$$V_{Loop}(r) = -\frac{10}{3\hbar}\frac{1}{2\pi r}Ze^2\alpha\ln(\theta\Lambda^2) - \frac{4\alpha}{15\hbar}Ze^2\lambda_e^2\delta^3(r)$$

# Energy Spectrum of NC Hydrogen atom

$$E_{nlj_z}^{NC} = -\left[\frac{20}{3\hbar}\frac{(Z\alpha)^2}{n^2 a_0}\ln(\theta\Lambda^2) + \frac{2\alpha^5 Z}{15\hbar}\lambda_e^2 m^3 + \right.$$

$$\left. + Z\alpha^2\,\theta\,\gamma_{Euler}\,j_z\left(3-\frac{2}{3}\right)\left(1\mp\frac{1}{2l+1}\right)\left(\frac{1}{n^3 l(l+1)(l+1/2)}\right)\right]$$

# Bound on the value of $q$

**For this purpose we have to consider only the shifting only at tree level**

**By comparing with available experimental data**

$$\theta \leq (10^4 GeV)^{-2}$$

# Contribution to Stark effect

$$\Delta E_{nlj_z}^{NCStark} = \frac{e}{2\hbar}\vec{\mathscr{E}} \times \vec{\theta}.\langle nl'jj'_z|\vec{P}|nljj_z\rangle$$

$$\Delta E_{nlj_z}^{NCStark} = 0$$

# Contribution to Zeeman effect

$$\Delta E_{OneLoop}^{NCZeeman} = -\frac{e}{2\ m_e\ c}\frac{\alpha\gamma_{Euler}m_e^2}{3\pi\hbar}\left(1 - F(q^2)\frac{m_p}{m_e}\right)\vec{\theta}.\vec{B}$$

# Thank You

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