Non-commutative field theories

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Introduction

\[ [x^i, x^j] = i \theta^{ij} \]

- Renormalizability at short distance
- Quantum Gravity
- String Theory
- UV-IR Mixing
Steps to develop any Q.F. theory

First get Langrangian from action integrand

Get Hamiltonian

Go to quantum mechanics
Why, don’t short-circuit

1:- Hamiltonian (difficult to make relativistic invariant)

2:- We have rule only for Lagrangian quadratic in velocity
Constraint on Hamiltonian formalism

1:- primary constraint

\[ \Phi_m(q, p) = 0 \]

because

\[ p = p(q) \]

only

i.e. then \( q \) & \( p \) will linearly independent

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_n} \right) = \left( \frac{\partial L}{\partial q_n} \right) \]

Important to note:
lagrangian is not completely arbitrary

Important to note

1=0 for simple case \( L=q \)

i.e. not consistence
So Hamiltonian “ $H$ ”

\[ H = p_n q_n^* - L \]

become

\[ H^* = H + c_m \Phi_m \text{ because } \Phi_m = 0 \]

$\Rightarrow$ Hamiltonian is not uniquely determine
\[
\{ \mathcal{A}, \mathcal{B} \}_{DB} = \{ \mathcal{A}, \mathcal{B} \} - \{ \mathcal{A}, X_s \} X_{ss'} \{ X_{s'}, \mathcal{B} \}
\]

\[
X_{ss'} = \Delta^{-1} = \begin{pmatrix}
\{ T^\alpha, T^\beta \} & \{ T^\alpha, G_\beta \} \\
\{ G_\alpha, T^\beta \} & \{ G_\alpha, G_\beta \}
\end{pmatrix}^{-1}
\]
Usual field theories

\[
\left\{ q^\alpha, q^\beta \right\}_{DB} = 0 , \\
\left\{ q^\alpha, p_\beta \right\}_{DB} = \delta^\alpha_\beta , \\
\left\{ p_\alpha, p_\beta \right\}_{DB} = 0 , \\
\left\{ q^\alpha, v_\beta \right\}_{DB} = \delta^\alpha_\beta , \\
\left\{ q^\alpha, \pi^\beta \right\}_{DB} = 0 .
\]
Non-commutative field theories

a), \[ \{q^\alpha, q^\beta\}_{DB} = -2 \theta^\alpha_\beta \]

b), \[ \{q^\alpha, p_\beta\}_{DB} = \delta^\alpha_\beta \]

c), \[ \{p_\alpha, p_\beta\}_{DB} = 0 \]

d), \[ \{q^\alpha, \pi_\beta\}_{DB} = \delta^\alpha_\beta \]

e), \[ \{q^\alpha, \pi^\beta\}_{DB} = -\theta^\alpha_\beta \]
Correspondence transformation

• By transforming the physical variables as

\[ q^\alpha \longrightarrow Q^\alpha \equiv q^\alpha + \theta^{\alpha\beta} p_\beta \]

\[ p^\alpha \longrightarrow P_\alpha \equiv p_\alpha \]

• By replacing all usual product with star *-product

\[ \psi (q) \star \phi (q) \equiv e^{i\theta^{\alpha\beta} \partial_{\delta} \partial_{\delta'} \psi (q + \xi) \phi (q + \zeta)} \bigg|_{\xi=\zeta=0} \]
Types of non-commutative field theories

1. General space-time non-commutative theories, i.e., $\theta_{0i} \neq 0$

2. Particular space non-commutative theories, i.e., $\theta_{0i} = 0$
CPT Symmetry

- Parity (P)

1. Parity transformation are same as usual QED for space-space non-commutativity

   but

2. For general space-time non-commutativity, Parity (P) is broken

   Because we have to replace $\theta_{0i}$ by $-\theta_{0i}$ along usual transformation

   but there should be no change in $\theta_{ij}$
Charge Conjugation

For both space-space and space-time non-commutivity

\[ \theta \rightarrow -\theta \]

along with usual transformation
Time reversal (T)

1. For space-space non-commutativity along usual Time reversal transformation, we have to replace
   \[ \theta \rightarrow -\theta \]

2. For general space-time non-commutativity, Time reversal (T) is broken
   Because we have to
   \[ \theta_{ij} \rightarrow -\theta_{ij} \]
   But there should no change in \[ \theta_{0i} \]
CP and CT Symmetry

- For Space-Space non-commutativity
  1. CP violate
  2. CT is invariant

- For Space-Time non-commutativity
  both CP and CT is broken as C, T and P are all separately broken
Feynman rules for NCQED

For each incoming electron: \( u_r(p) \)

For each outgoing electron: \( \bar{u}_r(p) \)

For each incoming positron: \( \bar{v}_r(p) \)

For each outgoing positron: \( v_r(p) \)
For each incoming photon: $\varepsilon_{r\mu}(k)$

For each outgoing photon: $\varepsilon_{\eta\mu}^*(k)$
1. Fermion-fermion-photon vertex:

\[ i e \gamma^\mu c \frac{i}{2} p_i \times p_f \]
2. Photon-photon-photon vertex:

\[
= -2e \sin \left( \frac{1}{2} k_1 \times k_2 \right) C^{\mu \nu \rho} (k_1, k_2, k_3)
\]

\[
= -2e \sin \left( \frac{1}{2} k_1 \times k_2 \right) \times \left[ (k_1 - k_2)^\rho g^{\mu \nu} + (k_2 - k_3)^\nu g^{\mu \rho} + (k_3 - k_1)^\nu g^{\mu \rho} \right]
\]
3. Photon-photon-photon-photon vertex:

\[ -4ie^2 \left[ (g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \times \sin \left( \frac{k_1 \times k_2}{2} \right) \sin \left( \frac{k_3 \times k_4}{2} \right) \\
+ (g^{\mu\sigma}g^{\nu\rho} - g^{\mu\nu}g^{\rho\sigma}) \times \sin \left( \frac{k_3 \times k_1}{2} \right) \sin \left( \frac{k_2 \times k_4}{2} \right) \\
+ (g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}) \times \sin \left( \frac{k_1 \times k_4}{2} \right) \sin \left( \frac{k_2 \times k_3}{2} \right) \right]. \]
4. Ghost-ghost-photon vertex

\[ p_i \quad \mu \quad k \quad p_f \]

\[ = 2iep_f^\mu \sin(\frac{1}{2}p_i \times p_f) \]
Radioactive correction

- One loop correction for electron self-energy

Figure 4.1: Electron self energy diagram

Figure 4.2: The diagram not possible
Photon self-energy

Figure 4.3: Photon self energy due to fermion loop correction

Figure 4.4: Photon self energy due to ghost-loop correction

Figure 4.5: Photon self energy due to photon loop correction with one vertex

Figure 4.7: Photon self energy due to photon loop correction with two vertex
Electron-Photon vertex correction

Figure 4.8: Vertex correction for QED like diagram

Figure 4.9: Vertex correction for QCD like diagram
UV/IR Mixing

\[ \Gamma^{\sigma(UV-ren)} = \frac{-\alpha e^{2p \times p'}}{\pi} \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 \delta(1 - \sum \alpha_i) \times \]

\[ \times \left( \frac{A^\sigma e^{-i(\alpha_2 + \alpha_3)p \cdot \tilde{q}}}{\alpha_1 m_\gamma^2 + (\alpha_2 + \alpha_3)^2 m^2 - \alpha_2 \alpha_3 q^2} + \right. \]

\[ \left. + \frac{\tilde{A}^\sigma (1 - e^{i(\alpha_2 + \alpha_3)p \cdot \tilde{q}} e^{-ip \times p'})}{m^2(\alpha_1 - \alpha_2 - \alpha_3) + m_\gamma^2(\alpha_2 + \alpha_3) + m^2(\alpha_2 + \alpha_3)^2 - \alpha_2 \alpha_3 q^2 -} \right. \]

\[ -2\gamma_{\text{Euler}} (B^\sigma e^{-i(\alpha_2 + \alpha_3)p \cdot \tilde{q}} + \tilde{B}^\sigma - \tilde{B}^\sigma e^{i(\alpha_2 + \alpha_3)p \cdot \tilde{q}} e^{-ip \times p'}) + \]

\[ + \Lambda^2_{\text{eff}} (C^\sigma e^{-i(\alpha_2 + \alpha_3)p \cdot \tilde{q}} - \tilde{C}^\sigma e^{i(\alpha_2 + \alpha_3)p \cdot \tilde{q}} e^{-ip \times p'}) \right) . \]  (4.2.66)
\[ C^\sigma = \frac{-\gamma^\sigma \tilde{q}.\tilde{q}}{8} + \frac{\tilde{q}^\sigma \gamma.\tilde{q}}{4} \]

and

\[ \tilde{C}^\sigma = \frac{\gamma.\tilde{q}q^\sigma}{4} + \frac{\gamma^\sigma \tilde{q}q}{8} \]

\[ \Lambda_{eff}^2 = \frac{1}{\Lambda^{-2} - \frac{\tilde{q}q}{4}}. \]
For UV limit
\[
\frac{1}{\Lambda^2} \ll \tilde{q}.\tilde{q} \Rightarrow \Lambda_{\text{eff}} \sim \frac{1}{\tilde{q}.\tilde{q}}
\]  
(Term is finite)

For IR limit
\[
\frac{1}{\Lambda^2} \gg \tilde{q}.\tilde{q} \Rightarrow \Lambda_{\text{eff}} \sim \Lambda^2
\]  
(Seems to lead IR divergence)

But \[
\Lambda^2 \tilde{q}^2 \ll 1 \]
in IR limit.

Therefore \[
C^\sigma \]
terms are irrelevant.  
(Term is again finite)
Photon deflection by coulomb field

For \( \theta_{01} = \theta_{03} = 0 \) and \( \theta_{13} = 1 \)

\[
\frac{d\sigma}{d\Omega} = \frac{Z^2 \alpha^2 E^2}{4 \lambda^4} \frac{(1 + 2 \sin^2(\phi/2))(1 - \sin^2(\phi/2))}{\sin^2(\phi/2)}
\]
Figure 4.11: The behavior of differential cross section with the scattering angle $\phi$ for scattered photon due to coulomb field with $Z = 1, \alpha = 1/128 \& \Lambda = 1500$ GeV with different values of $E = 400, 2000$ and 3000.
Interpretation to radioactive correction

● Vertex correction
Correction to the electron response to a given applied field

● Loop correction
Correction to the electro magnetic field itself
Interpretation of different terms of non-commutative vertex

Figure 5.1: Vertex correction

Figure 5.2: Vertex correction
\[ M = ie\bar{\nu}(p')\gamma^\mu u(p)\tilde{A}_\mu(q) \quad \text{In higher order} \quad M = ie\bar{\nu}(p')\Gamma^\mu u(p)\tilde{A}_\mu(q) \]

Where for usual QED

List of vectors: \( p^\mu, \ p'^\mu, \ \gamma^\mu \).  

List of scalars: \( q^2, \ m, \ e \).

Therefore we have

\[ \Gamma^\mu = F_1(q^2)\gamma^\mu + \frac{i\sigma^{\mu\nu}q^\nu}{2m}F_2(q^2) \]

At tree level \( F_1(q^2) = 1 \) and \( F_2(q^2) = 0 \).
In Coulomb gauge

\[ \widetilde{A}_u(q) = (\widetilde{A}_0, 0) \]

\[ \mathcal{M} = ie\widetilde{\phi}(q)F_1(0) \]

\[ V(x) = ie\widetilde{\phi}(q)F_1(0) \]

\( F_1(0) \) represents the electric charge of the electron
For static vector potential

\[ \tilde{A}_a(q) = (0, \tilde{A}_i) \]

\[ \mathcal{M} = ie \, 2m \, \xi^\dagger \frac{-i}{2m} \epsilon^{ijk} q_j \sigma_k \left[ F_1(0) + F_2(0) \right] \xi \tilde{A}_i(q) \]

\[ \tilde{B}^k(q) = -i \epsilon^{ijk} q_j \tilde{A}_i(q) \]

\[ \mathcal{M} = ie \, \xi^\dagger \sigma_k \left[ F_1(0) + F_2(0) \right] \xi \tilde{B}^k(q) \]

\[ V(x) = -\langle \mu \rangle \cdot B(x) \]

\[ \langle \mu \rangle = [F_1(0) + F_2(0)] \left( \frac{e}{2m} \right) \xi^\dagger \sigma \xi \]
Vertex in non-commutative field theory

\[ \Gamma^\mu \quad \text{at tree level} \quad e^{i p \cdot \tilde{q}^\lambda} \gamma^\mu \]

List of scalars: \( m, e, q^2, \tilde{q} \cdot \tilde{q}, \gamma \cdot \tilde{q}, p \cdot \tilde{q} \)  
List of vectors: \( \gamma^\mu, (p' + p)^\mu, q^\mu, \tilde{q}^\mu \)

\[ \Gamma^\mu = A \gamma^\mu + B (p' + p)^\mu + C q^\mu + D \tilde{q}^\mu \]

\[ \Gamma^\mu = F_1( q^2, p \cdot \tilde{q}, \gamma \cdot \tilde{q}) \gamma^\mu + \frac{i \sigma^{\mu \nu} q_\nu}{2m} F_2( q^2, p \cdot \tilde{q}, \gamma \cdot \tilde{q}) + D( q^2, p \cdot \tilde{q}, \gamma \cdot \tilde{q}) \tilde{q}^\mu \]

\[ \Gamma^\mu = F_1(a) \gamma^\mu + (F_1(b) p \cdot \tilde{q}) \gamma^\mu + \frac{i \sigma^{\mu \nu} q_\nu}{2m} \left[ F_2(a) + (F_2(c) \gamma \cdot \tilde{q}) \right] + D(a) \tilde{q}^\mu \]
The term containing $F_{1(a)} \gamma^{\mu}$

$$F_{1(a)}(q^2) = \frac{-\alpha e^{\frac{i}{2}p \cdot p'}}{\pi} \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 \delta(1 - \sum \alpha_i) \times \left( 1 - e^{i(\alpha_2 + \alpha_3) p \cdot \hat{q} e^{-ip \cdot p'}} \right)$$

$$\times \left\{ \left[ \frac{(2p' \cdot p - (\alpha_2 + \alpha_3)(p' + p)^2 + m^2(\alpha_2 + \alpha_3)^2 - \alpha_2 \alpha_3 q^2)}{2(\alpha_1 m^2_{\gamma} + (\alpha_2 + \alpha_3)^2 m^2_\gamma - \alpha_2 \alpha_3 q^2)} + \gamma_{Euler} \right] e^{-i(\alpha_2 + \alpha_3) p \cdot \hat{q}} 
+ \left[ \frac{((\alpha_2 + \alpha_3)(p' + p)^2 - 3m^2 - m^2(\alpha_2 + \alpha_3)^2 + \alpha_2 \alpha_3 q^2)}{m^2(\alpha_1 - \alpha_2 - \alpha_3) + m^2_{\gamma}(\alpha_2 + \alpha_3) + m^2(\alpha_2 + \alpha_3)^2 - \alpha_2 \alpha_3 q^2} - \frac{3 \gamma_{Euler}}{2} \right] \right\}.$$
The term containing \((F_{1(b)} p \cdot \bar{q}) \gamma^\mu\)

\[
F_{1(b)} = \frac{-\alpha e^{\frac{i}{2} p \times p'}}{\pi} (i \gamma_{Euler}) \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 \delta(1 - \sum \alpha_i) \left(1 - e^{-i p \times p'}\right) (2 - \alpha_2 - \alpha_3) e^{-i(\alpha_2 + \alpha_3) p \cdot \bar{q}}
\]
The term containing $\frac{i\sigma^{\mu\nu} q_{\nu}}{2m} F_{2(a)}$

\[ F_{2(a)} = -\alpha e^{i p \cdot p^\prime} \frac{1}{\pi} \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 \delta(1 - \sum \alpha_i) \times \]

\[ \left\{ \frac{m\alpha_1(\alpha_2 + \alpha_3)e^{i(\alpha_2 + \alpha_3)p \cdot \bar{q}}}{\alpha_1 m_\gamma^2 + (\alpha_2 + \alpha_3)^2 m^2 - \alpha_2 \alpha_3 q^2} + \frac{m\alpha_1(\alpha_2 + \alpha_3)(1 - e^{i(\alpha_2 + \alpha_3)p \cdot \bar{q}} e^{-ip \cdot p^\prime})}{m^2(\alpha_1 - \alpha_2 - \alpha_3) + m_\gamma^2(\alpha_2 + \alpha_3) + m^2(\alpha_2 + \alpha_3)^2 - \alpha_2 \alpha_3 q^2} \right\} \]
The term containing $\frac{i\sigma^{\mu\nu}q_{\nu}}{2m}(F_{2(c)}\gamma_5\tilde{q})$

$$F_{2(c)} = \frac{-\alpha e^{\frac{i}{2}p \times p'}}{\pi} \frac{i\gamma_{Euler}}{2} \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 \delta(1 - \sum \alpha_i) \times$$

$$\left((2 - \alpha_2 - \alpha_3) e^{-i(\alpha_2 + \alpha_3)p \cdot \tilde{q}} + (1 + \alpha_2 + \alpha_3) e^{i(\alpha_2 + \alpha_3)p \cdot \tilde{q}} e^{-i p \times p'}\right)$$

This term gives electric dipole moment of the form

$$\langle P \rangle_i = 2i F_{2(c)} \vec{p}_i$$

The term containing $D_{(a)}\tilde{q}^\mu$

$$D_{(a)} = -\alpha e^{\frac{i}{2}\pi p \times p'} \frac{i}{\pi} m \gamma_{\text{Euler}} \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 \delta(1 - \sum \alpha_i) \times$$

$$\times \left( (1 + \alpha_2 + \alpha_3) e^{-i(\alpha_2 + \alpha_3) p \cdot \tilde{q}} - (2 - \alpha_2 - \alpha_3) e^{i(\alpha_2 + \alpha_3) p \cdot \tilde{q}} e^{-ip \times p'} \right)$$

This term gives magnetic moment of the form

$$\langle \mu \rangle = \frac{D_{(a)}}{i} \vec{\theta}, \quad \text{with} \quad \theta_i \equiv \epsilon_{ijk}\theta_{jk}$$

Therefore it is straightforward to see that the magnetic moment due to this term does not depend upon the spin. It is pure non-commutative aspect.


We have observed that magnetic moment in non-commutative electron photon vertex has two parts

1. One spin dependent part,

2. Other is spin independent part, which is proportional to $\theta^{\mu\nu}$ and have lost contribution in the limit $\theta \to 0$.  

Loop correction
(The Lamb Shift in NCQED)

- Hydrogen atom in NCQM (at tree level)

\[ \Delta E = -\frac{m_e c^2}{4} (Z\alpha)^4 \frac{\theta}{\lambda_e^2} j_z \left( 1 + \frac{1}{2l+1} \right) \left( \frac{1}{n^3 l(l+1)(l+1/2)} \right) \delta_{j'j} \delta_{l'lj} \]

For

\[ n^{j_z} l_j \]

Polarized lamb-shift
One loop correction

- Vertex correction (two different types of diagram)

\[ V_{\text{Vertex}} = -\frac{Ze^2}{4\pi} \alpha \gamma_{\text{Euler}} \left( 3 - \frac{2}{3} \right) \frac{L_\theta}{\hbar r^3} \]

- Loop correction (four different types of diagram)

\[ V_{\text{Loop}}(r) = -Ze^2 \alpha \frac{10}{3\hbar} \int d^3q \frac{1}{\bar{q}^2} e^{-i\bar{q} \cdot \bar{r}/\hbar} \left[ \ln \left( \frac{\bar{q}^2 \tilde{q}^2}{\hbar^4} \right) + \frac{2}{25} \frac{\bar{q}^2}{m^2} \right] \]

\[ V_{\text{Loop}}(r) = -\frac{10}{3\hbar 2\pi r} Ze^2 \alpha \ln(\theta \Lambda^2) - \frac{4\alpha}{15\hbar} Ze^2 \lambda^2 \delta^3(r) \]
Energy Spectrum of NC Hydrogen atom

\[ E_{nljz}^{NC} = - \left[ \frac{20}{3\hbar} \frac{(Z\alpha)^2}{n^2a_0} \ln(\theta \Lambda^2) + \frac{2\alpha^5 Z}{15\hbar} \chi_e^2 m^3 + \right. \\
+ Z\alpha^2 \theta \gamma_{Euler} j_z \left( 3 - \frac{2}{3} \right) \left( 1 + \frac{1}{2l + 1} \right) \left( \frac{1}{n^3l(l+1)(l+1/2)} \right) \]
Bound on the value of $\theta$

For this purpose we have to consider only the shifting only at tree level

By comparing with available experimental data

$$\theta \leq (10^4 GeV)^{-2}$$
Contribution to Stark effect

$$\Delta E^{NC\text{Stark}}_{nljz} = \frac{e}{2\hbar} \vec{\mathcal{E}} \times \vec{\theta} \langle nl'jj'_z |\vec{P}|nljj_z\rangle$$

$$\Delta E^{NC\text{Stark}}_{nljz} = 0$$

Contribution to Zeeman effect

$$\Delta E^{NC\text{Zeeman}}_{\text{OneLoop}} = -\frac{e}{2m_e c} \frac{\alpha \gamma_{\text{Euler}} m_e^2}{3\pi \hbar} \left(1 - F(q^2) \frac{m_p}{m_e}\right) \vec{\theta} \cdot \vec{B}$$
Thank You