A novel Approach to Quantum Plasma

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White dwarf

A white dwarf-a degenerate dwarf is a small star and its volume is comparable to that of the Earth, the mass is comparable to that of sun. They are composed of carbon and oxygen. Over a very long time, a white dwarf will cool to temperatures at which it will no longer be visible, and become a cold black dwarf.

The number density

$$n \cong 10^{30} / cm^{3}$$

The temperatures extend from 150.000K to 4.000 K

Magnetic fields have been discovered in well over 100 white dwarfs, ranging from 2×10^{3} to 10^{9} gauss.

The relationship between mass and radius

$$R \sim \frac{1}{M^{1/3}}$$

S. Chandrasekhar limit (1931)

Pulsar

When the volume per atom becomes less than the usual size of the atom, the atoms lose their individuality, and so the substance is transformed into a highly compressed plasma of electrons and nuclei. In this case a medium becomes a degenerate Fermi gas, and nuclear reactions consisting in the capture of electrons by nuclei decreases the charge on the nucleus, so

$$p + e^- \rightarrow n + \nu$$

Pulsar is the neutron star in which 99% electrons have been captured by protons. 1% are electrons and protons.

The density of neutrons $n \cong 10^{38} / \mathrm{cm}^3$

For electrons we have $n_e \sim n_p \cong 10^{36} / \mathrm{cm}^3$,

which means they are in strongly degenerate state.

The radius of a neutron star ~ 10^6 cm

The magnetic field $\cong 10^{11} - 10^{13}$ gauss

The pulsar is a radio, optical, X-ray and gamma-emitting neutron star associated with Supernova Remnant.

Another application of the quantum theory is in a nanotechnology, where size of natural and artificial structures is the nanometer scale, i.e., in the range of from $1 \mu\mathrm{m}$ down to $10 \mathrm{A}^0$.

Quantum Particles: e^-, e^+, light ions as proton, D^+, He^3

and He^4 at a high density and low temperatures. In

semiconductors with a large number of light carriers

(electrons), $n_e \geq 10^{16} \rightarrow 10^{18}\, cm^{-3}$, a mass $m_e^* \approx 10^{-2} m_e$

and $T_F < 10^2\, K$. The degeneracy of heavy charge carriers (holes) occurs at lower temperatures.

For electrons: $T_F^e \sim 5\times10^3\,K$

For ions: $T_F^i \sim \dfrac{m_e}{m_p} 5\times10^3\,K$

For proton gas : $T_F^p \sim 2.6\,K$

For dusty compounds: $T_F^D \sim \dfrac{m_e}{m_D} 5\times10^3\,K = 5\times10^{-8}\,K$

If $m_D = 10^{-16}\,g$ and $n_D \sim 10^{22}\,cm^{-3}$

In a pure state the Wigner distribution function

$$f^{w}(\vec{r},\vec{p},t)=\frac{1}{(2\pi\hbar)^{3}}\int d\vec{x}\,\Psi^{*}(\vec{r}+\vec{x}/2)\Psi(\vec{r}-\vec{x}/2)e^{i\vec{p}\cdot\vec{x}/\hbar}$$

and

$$\left[\frac{\partial}{\partial t}+\frac{\vec{p}}{m}\cdot\frac{\partial}{\partial\vec{r}}-\frac{2}{\hbar}\sin\left(\frac{\hbar}{2}\frac{\partial}{\partial\vec{p}}\cdot\frac{\partial}{\partial\vec{r}}\right)U\right]f=0$$

$$\sin(y)\approx y$$

$$\frac{\partial f}{\partial t}+\left(\vec{v}\cdot\vec{\nabla}\right)f-\frac{\partial U}{\partial\vec{r}}\cdot\frac{\partial f}{\partial\vec{p}}=0$$

$\left|\Psi(\vec{r},t)\right|^{2}\sim$ this is a probability density.

$$\int d\vec{r}\,\left|\Psi\right|^{2}=1$$

- Condition for quasi-classical motion is that the particle de Broglie wavelength $\hbar / P_F$ be small compared with the characteristic length *L* over which the density varies considerably,

$$\frac{\hbar}{p_F} << L \sim \frac{1}{k} \;, \quad \hbar k << p_F \sim \hbar\, n^{\frac{1}{3}} \;\text{ or }\; \lambda >> n^{-\frac{1}{3}}$$

- The same we obtain from $\varepsilon_F >> \hbar\omega$
- In the fluid equation, we have two terms

$$-\frac{1}{n}\nabla p_F + \frac{\hbar^2}{2m}\nabla \frac{1}{\sqrt{n}}\Delta\sqrt{n}$$

- From this expression follows that the first term is much greater than the second one, as $\lambda >> \dfrac{1}{n^{1/3}}$

The Non-relativistic Pauli equation

$$i\hbar\frac{\partial\Psi_\alpha}{\partial t}+\frac{\hbar^2}{2m_\alpha}\Delta\Psi_\alpha-\left[\frac{ie\hbar}{2m_\alpha c}(\vec{A}\cdot\nabla+\nabla\vec{A})+\frac{e^2A^2}{2m_\alpha c^2}+e_\alpha\varphi-\vec{\mu}_\alpha\cdot\vec{H}\right]\Psi_\alpha=0 \quad \textbf{(1)}$$

$$\vec{\mu}_\alpha=\frac{e\hbar}{2m_\alpha c}\vec{\sigma}=\mu_\beta\vec{\sigma} \quad \textbf{(2)}$$

where μ_β is the Bohr magneton and $\vec{\sigma}$ is the operator of the single particle
Use of the Madelung representation of the complex function Ψ_α

$$\Psi_\alpha(\vec{r},t,\vec{\sigma})=a_\alpha(\vec{r},t,\vec{\sigma})\exp\frac{iS_\alpha(\vec{r},t,\vec{\sigma})}{\hbar} \quad \textbf{(3)}$$

where $a_\alpha(\vec{r},t,\vec{\sigma})$ and $S_\alpha(\vec{r},t,\vec{\sigma})$ are real, in the Pauli equation (1),
yields the following two equations

$$\frac{\partial a_\alpha^2(\vec{r},t,\vec{\sigma})}{\partial t}+\nabla\cdot\left(a_\alpha^2(\vec{r},t,\vec{\sigma})\frac{\vec{p}_\alpha}{m_\alpha}\right)=0 \quad \textbf{(4)}$$

$$\frac{d\vec{p}_\alpha}{dt}=e_\alpha\left(\vec{E}+\frac{\vec{v}_\alpha\times\vec{H}}{c}\right)+\frac{\hbar^2}{2m_\alpha}\nabla\frac{1}{a_\alpha}\nabla^2a_\alpha+\mu_\beta\nabla(\vec{\sigma}\cdot\vec{H}) \quad \textbf{(5)}$$

At ($\vec{E}=0$, $\vec{H}=0$), after the linearization of Eqs. (4) and (5), we get the frequency of quantum oscillations of a free electron

$$\omega_q=\frac{\hbar k^2}{2m} \quad \textbf{(6)}$$

We introduce a density of probability distribution f_S in phase space for the single particle

$$|\Psi|^2 = \int d^3p \; f_S(\vec{r}, \vec{p}, t) \qquad (7)$$

This function f_S, obviously must satisfy the normalization condition over all phase space

$$\int d^3r \int d^3p \; f_S = \int d^3r \; |\Psi|^2 = 1 \qquad (8)$$

Moreover, Bogolyubov has introduced a one particle distribution function for the system as a whole from Liouville's theorem regarding the distribution function $f_\alpha^N(t, \tau_1, \tau_2 ... \tau_N)$, (where τ_α is the set of coordinates and momentum components for the α particle), and derived the Vlasov and Boltzmann equations in the gas approximation, which means that the plasma parameter η (representing the ratio of the average potential energy $< U >$ of particles interaction to the average kinetic energy $< \varepsilon_k >$) must be less than unity, i.e., $\eta = \frac{<U>}{<\varepsilon_k>} \ll 1$. Note that the one particle distribution function $f(\vec{r}, \vec{p}, t)$ is normalized to unity, whereas the Liouville's function f^N to total number of particles, i.e., $f^N = N f(\vec{r}, \vec{p}, t)$. The same relation between f^N and $f(\vec{r}, \vec{p}, t)$ in an alternative description of kinetic theory was obtained by Klimontovitch. To make it more lucid , we shall give a simple explanation about a single particle and one particle distribution function . Namely, the probability density of the single particle is one particle per unit volume, $n^s(\vec{r}, t) = |\Psi(\vec{r}, t)|^2$, with dimensions 1/V. Whereas the one particle distribution function means that in spite of the large number of particles in the unit volume all of them have only one $\vec{r}$ and $\vec{p}$. This permits us to express the number density of particles per unit volume as $n(\vec{r}, t) = \int d^3p \; f(\vec{r}, \vec{p}, t) = N/V$. Therefore, we can write

$$n(\vec{r}, t) = N n^s(\vec{r}, t) = N |\Psi(\vec{r}, t)|^2 = \int d^3p \; f(\vec{r}, \vec{p}, t) \qquad (9)$$

Thus, $n(\vec{r}, t)$ is the density of quantum particles per unit volume. We have assumed that the total number of particles of each species is conserved.

$$\frac{\partial f_\alpha(\vec{r},\vec{p},t,\vec{\sigma})}{\partial t} + \left(\frac{\partial \vec{r}}{\partial t}\cdot\nabla\right) f_\alpha(\vec{r},\vec{p},t,\vec{\sigma}) + \frac{d\vec{p}_\alpha}{dt}\frac{\partial f_\alpha(\vec{r},\vec{p},t,\vec{\sigma})}{\partial \vec{p}} = C(f_\alpha)\ . \quad \textbf{(10)}$$

When the spin of particles is taken into account, the distribution function f_α becomes an operator with respect to the spin variables σ. In this case the total number density of particles $n_\alpha(\vec{r},t)$ equals

$$n_\alpha(\vec{r},t) = \sum_\sigma \int \frac{d^3p}{(2\pi\hbar)^3} f_\alpha(\vec{r},\vec{p},t,\vec{\sigma})\ . \quad \textbf{(11)}$$

Hereafter, we consider the system as a spinless. We substitute the equation of motion of single particle (5) (neglecting the last term) into the kinetic equation (10) taking into account the definition of the density of particles (9) to obtain

$$\frac{\partial f_\alpha}{\partial t} + (\vec{v}\cdot\nabla) f_\alpha + e_\alpha\left(\vec{E} + \frac{\vec{v}_\alpha \times \vec{H}}{c}\right)\frac{\partial f_\alpha}{\partial \vec{p}} + \frac{\hbar^2}{2m_\alpha}\nabla\frac{1}{\sqrt{n_\alpha}}\Delta\sqrt{n_\alpha}\,\frac{\partial f_\alpha}{\partial \vec{p}} = C(f_\alpha)\ .$$

(12)

Using the Poisson's equation

$$\Delta\delta\varphi = 4\pi e\left\{2\int \frac{d\vec{p}}{(2\pi\hbar)^3}\delta f_e - 2\int \frac{d\vec{p}}{(2\pi\hbar)^3}\delta f_i\right\} \quad \textbf{(13)}$$

and assuming the Fermi degeneracy temperature $T_F = \frac{\varepsilon_F}{K_B}$ (K_B is the Boltzmann coefficient, the Fermi distribution function is the step function $f_{\alpha 0} = \Theta(\varepsilon_{F\alpha} - \varepsilon)$, where $\varepsilon_{F\alpha} = \frac{m_\alpha v_{F\alpha}^2}{2}$) much more than the Fermi gas temperature, then we obtain after some algebra the quantum dispersion equation

$$\varepsilon = 1 + \sum_\alpha \frac{3\omega_{p\alpha}^2}{\Gamma_\alpha k^2 v_{F\alpha}^2}\left\{1 - \frac{\omega}{2kv_{F\alpha}}\ln\frac{\omega + kv_{F\alpha}}{\omega - kv_{F\alpha}}\right\} = 0 \quad \textbf{(14)}$$

where

$$\Gamma_\alpha = 1 + \frac{3\hbar^2 k^2}{4m_\alpha v_{F\alpha}^2}\left(1 - \frac{\omega}{2kv_{F\alpha}}\ln\frac{\omega + kv_{F\alpha}}{\omega - kv_{F\alpha}}\right)$$

Let us first consider the electron Langmuir waves, supposing that the ion mass $m_i \to \infty$ and $\omega \gg kv_{Fe}$, or the range of fast waves, when the phase velocity exceeds the Fermi velocity of electrons. In this case we get the dispersion relation

$$\omega^2 = \omega_{pe}^2 + \frac{3k^2 v_{Fe}^2}{5} + \frac{\hbar^2 k^4}{4m_e^2} + \ldots \quad \textbf{(15)}$$

which has been previously derived by Klimontovich and Silin

As k increases, Eq.(15) becomes invalid, but still $\omega > kV_{Fe}$ and the Landau damping is absent. We now introduce the Thomas-Fermi screening wave vector $k_{TF} = \frac{\sqrt{3}\omega_{pe}}{v_{Fe}}$. In the limit $k^2 \gg k_{FT}^2$, ω tends to kv_{Fe} (at $m_i \to \infty$) and we obtain from **(14)**

$$\omega = kv_{Fe}\Big(1 + 2\exp\{-\frac{2(\frac{k^2}{k_{FT}^2} + \frac{\omega_q^2}{\omega_{pe}^2})}{1 + \frac{\omega_q^2}{\omega_{pe}^2}}\}\Big) \quad \textbf{(16)}$$

If we neglect the quantum term ω_q in Eq.(16), then we recover waves known as the zero sound

Special and very important case in a quantum plasma is an one-fluid approximation. We further assume that the quasi neutrality

$$n_e = n_i \quad \textbf{(17)}$$

is satisfied. This equation along with the equation of motion of ions and the equation giving the adiabatic distribution of electrons allow us to define the potential field. In such approximation the charge is completely eliminated from the equations, and the Thomas-Fermi length $r_{FT} = \frac{v_{Fe}}{\sqrt{3}\omega_{pe}}$ disappears with it.

In order to construct the one-fluid quantum kinetic equation, we neglect the time derivative in the equation (12) of electrons, as well as the collision terms, suppose $\vec{E} = -\nabla\varphi$ and $\vec{H} = 0$, and write the dynamic equations for the quasi-neutral plasma (17)

$$(v \cdot \nabla) f_e + \nabla\Big(e\varphi + \frac{\hbar^2}{2m_e}\frac{1}{\sqrt{n}}\Delta\sqrt{n}\Big)\frac{\partial f_e}{\partial p} = 0 \quad \textbf{(18)}$$

$$\frac{\partial f_i}{\partial t} + (v \cdot \nabla) f_i - \nabla e\varphi \, \frac{\partial f_i}{\partial p} = 0 \qquad (19)$$

In Eq.(19) we have neglected the quantum term as a small one. Note that the Fermi distribution function of electrons

$$f_e = \frac{1}{\exp\left\{ \frac{\frac{p^2}{2m_e} - U - \mu_e}{T} \right\} + 1} \qquad (20)$$

satisfies the equation (18). Here $U = e\varphi + \frac{\hbar^2}{2m_e}\frac{1}{\sqrt{n}}\Delta\sqrt{n}$, and μ_e is the chemical potential.

For the strongly degenerate electrons, i.e., $T_e \to 0$ ($\mu_e = \varepsilon_F$), the Fermi distribution function becomes the step function

$$f_e = \Theta(\varepsilon_F + U - \frac{p^2}{2m_e}) \; , \qquad (21)$$

which allows us to define the density of electrons ($n_e = n_i = n = 2\int \frac{d\vec{p}}{(2\pi\hbar)^3} f$)

$$n = \frac{p_{Fe}^3}{2\pi^2\hbar^3} \left(1 + \frac{e\varphi + \frac{\hbar^2}{2m_e}\frac{1}{\sqrt{n}}\Delta\sqrt{n}}{\varepsilon_{Fe}} \right)^{3/2} \qquad (22)$$

We now express $e\varphi$ from the equation (22) and substitute it into the kinetic equation (19) to obtain

$$\frac{\partial f}{\partial t} + (v \cdot \nabla) f - \nabla \left\{ \varepsilon_{Fe} \left(\frac{n}{n_0} \right)^{2/3} - \frac{\hbar^2}{2m_e} \frac{1}{\sqrt{n}} \Delta \sqrt{n} \right\} \frac{\partial f}{\partial p} = 0 \ . \qquad \textbf{(23)}$$

This is the nonlinear kinetic equation of quantum plasma in the one-fluid approximation, which incorporates the potential energy due to degeneracy of the plasma and the Madelung potential.

To consider the propagation of small perturbations $f = f_0(\vec{p}) + \delta f(\vec{r}, \vec{p}, t)$ and $n = n_0 + \delta n(\vec{r}, t)$, we shall linearize Eq.(23) with respect to the perturbations, look for a plane wave solution as $e^{i(\vec{k}\cdot\vec{r} - \omega t)}$, and derive the dispersion equation, which resembles the Bogolyubov's dispersion relation in the frequency range $kv_{Fe} \gg \omega \gg kv_{Fi}$,

$$\omega = k\sqrt{\frac{p_F^2}{3m_e m_i} + \frac{\hbar^2 k^2}{4m_i m_e}} = k\sqrt{\frac{2\varepsilon_{Fe}}{3m_i} + \frac{\hbar^2 k^2}{4m_i m_e}} \qquad \textbf{(24)}$$

We specifically note here that this type of spectrum (24) was derived by Bogolyubov for the elementary excitations in a quantum Bose liquid, and created the microscopic theory of the superfluidity of liqud helium.

How one can explain the similar spectrum (24) in the Fermi gas? In the dense and low temperature plasma the formation of bound states is possible due to the attractive character of the Coulomb force . As is well known, when the density of particles increases and the temperature goes to zero, the nuclear reaction leads to the capture of electrons by nuclei. In such reaction the charge on the ions (nucleus) decreases. Because we have assumed the quasi-neutrality, we have therefore supposed that all electrons are in the bound state with ions. Thus one can say that in the one-fluid approximation the Fermi plasma may become the Bose system due to the bound state or this approximation may imply the formation of the Bose atoms.

Following the standard method, we can derive the equations of continuity and motion of macroscopic quantities from Eq.(23)

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\vec{u}) = 0 \qquad \textbf{(26)}$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\frac{K_B T_{Fe}}{m_i} \nabla (\frac{n}{n_0})^{2/3} + \frac{\hbar^2}{2m_e m_i} \nabla \frac{1}{\sqrt{n}} \Delta \sqrt{n}\ , \qquad \textbf{(27)}$$

where $u(\vec{r}, t)$ is the macroscopic velocity of the plasma

$$\vec{u} = \frac{1}{n} \int \frac{2d\,\vec{p}}{(2\pi\hbar)^3} v f(\vec{r}, \vec{p}, t) \qquad \textbf{(28)}$$

The more general set of fluid equations for α kind of particles we can obtain from the equation (12) taking into account that any elastic scattering should fulfill the general conservation laws of particle number, momentum and energy

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \vec{u}_\alpha) = 0 \ , \qquad \textbf{(29)}$$

$$\frac{\partial < \vec{p}_\alpha >}{\partial t} + (\vec{u}_\alpha \cdot \nabla) < \vec{p}_\alpha > = e_\alpha \left(\vec{E} + \frac{1}{c} \vec{u}_\alpha \times \vec{B} \right) - \frac{1}{n_\alpha} \nabla P_\alpha + \frac{\hbar^2}{2m_\alpha} \nabla \frac{1}{\sqrt{n_\alpha}} \Delta \sqrt{n_\alpha}$$
$$+ \frac{1}{n_\alpha} \int \frac{2d\vec{p}}{(2\pi\hbar)^3} \, p_\alpha \, C(f_\alpha) \ , \qquad \textbf{(30)}$$

where $P_\alpha = \frac{1}{3m_\alpha} \int \frac{d\vec{p}}{(2\pi\hbar)^3} \frac{p_\alpha^2}{exp\{\frac{\varepsilon_\alpha - \mu_\alpha}{T_\alpha}\}+1}$.

Moreover, the equation of state of a degenerate Fermi plasma can be derived from the kinetic equation (12) by multiplying it by $\frac{p_\alpha^2}{2m_\alpha}$, integrating over the momentum and employing the equations of continuity (29) and motion (30). In the non-relativistic limit the result is

$$\frac{d}{dt} \ln \frac{< \varepsilon_\alpha >}{n_\alpha^{2/3}} = \frac{1}{n_\alpha < \varepsilon_\alpha >} \int \frac{2d\vec{p}}{(2\pi\hbar)^3} \, \varepsilon_\alpha \, C(f_\alpha) \ , \qquad \textbf{(31)}$$

where $\varepsilon_\alpha = \frac{(p_\alpha - < p_\alpha >)^2}{2m_\alpha}$ and $< \varepsilon_\alpha > = \frac{1}{n_\alpha} \int \frac{2d\vec{p}}{(2\pi\hbar)^3} \, \varepsilon_\alpha f_\alpha$ is the internal energy of particles, and the collision terms can be positive or negative.

Thanks