NONCOMMUTATIVE GEOMETRY:
Fuzzy spaces, the Groenewold-Moyal Plane

THIS IS A LEAD TALK FOR WHAT IS TO COME.

A TASTE OF THE FUTURE, ITS INTIMATIONS WITH A PERSONAL FOCUS.

IT IS WELL-KNOWN THAT MULTANI SOHAN HALWA DOES NOT EXIST OUTSIDE LAHORE JUST AS THERE IS NO PIZZA OUTSIDE NAPOLI.

MY TALK IS AT BEST A FORETASTE OF SUCH DELICIOUS FARE WHICH WAITS YOU.
1. Groenewold–Moyal Quantum Theory on the Moyal Plane, Fuzzy Spaces

Noncommutative geometry is a branch of mathematics due to Connes, Rieffel and others. Physicists in a very short time vulgarised it and use this phrase whenever spacetime algebra is noncommutative.

There are 2 such active fields in Physics at present:

1. Fuzzy Physics.
2. QFT (Quantum Field Theory) on a Groenewold–Moyal Plane.

Item 1 is evolving into a tool for to regulate QFT’s, and numerical work.
It is an alternative to lattice methods.

Item 2 is more a probe of Planck-scale physics.

This introductory talk will focus on items 1 & 2.
The Groenewold-Moyal plane has also emerged from string physics, as also fuzzy spaces, each being also a simple noncommutative model, merits study.

The Groenewold-Moyal plane first occurs in a letter from Heisenberg to Peierls. He suggests that the uncertainty principle \( \Delta x_m \Delta x_\nu \geq \frac{1}{2} |\Theta_{m\nu}| \) will provide short distance cut-off & regulate QFT's. He complains about his ignorance of mathematics to study this possibility.

Peierls communicates this idea to Pauli, and Pauli to Snyder.

Snyder writes first paper on the subject. This followed by a paper of Yang. Joe Weinberg, Syracuse, the "other" man in Oppenheimer case.
In mid-90's, Doplicher, Fredenhagen and Roberts systematically constructed unitary QFT's on Moyal plane, even with time-space noncommutativity.

Later string physics encountered these structures.

What is noncommutative geometry?

According to Connes, noncommutative geometry is a spectral triple, "Holy Trinity", $(\mathcal{A}, \mathcal{D}, \mathcal{H})$

$\mathcal{A} = \text{a C}^*\text{-algebra, possibly noncommutative}$

$\mathcal{D} = \text{Dirac operator}$

$\mathcal{H} = \text{a Hilbert space on which they are represented}$.

If $\mathcal{A}$ is commutative, we can recover a Hausdorff topological space on which $\mathcal{A}$ are functions, its differentiable structure,...
(Gel'fand-NAZMARK, Connes, ...) (There are additional axioms, due to Connes)

A class of examples with a noncommutative is due to Connes and Landi.

Many examples if some axioms not enforced: $SU(2)_q$, fuzzy spaces, Groenewold-Moyal plane, ...

Conceptual revolution: Manifolds are being replaced by their "duals," algebras, much as in quantum mechanics.

The Tangled Web

QFT Regularisation \[\rightarrow\] Fuzzy Physics

Hall Effect \[\rightarrow\] Groenevold-Moyal Plane

Quantum Gravity \[\rightarrow\] Strings D-Branes
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WHAT IS FUZZY PHYSICS?

2 DIMENSIONAL EXAMPLE: $S^2$

WE QUANTISE $S^2$ TO REGULARISE,

INTRODUCE SHORT DISTANCE CUT-OFF.

THUS: $S^2: [\mathbf{x} \in \mathbb{R}^3: \mathbf{x} \cdot \mathbf{x} = r^2]$

NOW CONSIDER ANGULAR MOMENTA \( L_i \):

\[
[L_i, L_j] = i \varepsilon_{ij} L_k; \quad L^2 = \ell (\ell + 1).
\]

SET

\[
\hat{x}_i = \frac{R L_i}{[\ell (\ell + 1)]^{1/2}} \Rightarrow
\]

\[
\hat{x} \cdot \hat{x} = R^2, \quad [\hat{x}_i, \hat{x}_j] = \frac{R}{[\ell (\ell + 1)]^{1/2}} i \varepsilon_{ij} \hat{x}_k.
\]

\( \hat{x}_i \in \text{Mat}_{2\ell+1} \). AS \( \ell \to \infty \), THEY BECOME COMMUTATIVE. THEY GIVE $S^2$, RADIUS $R$, FUZZY SPHERE \( S_{2\ell+1} \) OF DIMENSION $2\ell + 1$. 
Thus, in classical mechanics, the number of states in phase space volume $\Delta V = \int d^3p \, d^3q$ is infinite.

But we know since Planck and Bose that on quantisation, it becomes

$$\frac{\Delta V}{\hbar^3} = \text{finite}.$$ 

This is the idea.
WHY IS THIS SPACE FUZZY?

AS $\hat{x}_i, \hat{x}_j (1 \neq j)$ DO NOT COMMUTE

WE CANNOT SHARPLY LOCALISE $\hat{x}_i$.

ROUGHLY IN VOLUME $\sim 4\pi R^2$, THERE ARE $(2\ell + 1)$ STATES OR $\#$ STATES PER UNIT VOLUME $= (2\ell + 1) / (4\pi R^2)$. 
Scalar field $\Phi = \text{polynomial in } \chi_l$

$= (2l+1)$-dimensional matrices.

Differentiation: $d_x \Phi = [L_4, \Phi]$.

Infinitesimal rotation.

A scalar field action:

$$S = \lambda_1 Tr [d_x \Phi]^+ [d_x \Phi] + \lambda_2 Tr (\Phi^+ \Phi)^2$$

Simulations: X. Martin, Medina et al.

Major findings:

- Continuum limit exists (2nd order phase transition)
- If $\Phi = \sum c_{\ell m} \hat{y}_m$, $\hat{y}_m$ = spherical tensors, 3 phases:
  - Disordered: $\langle \sum l c_{\ell m}^2 \rangle = 0$
  - Uniform ordered: $\langle \sum l c_{\ell m}^2 \rangle \neq 0$, $\langle l c_{\ell m}^2 \rangle \neq 0$. \\

$\ell \neq 0$. \\

$\lambda_1$ and $\lambda_2$. \\

$Tr$: trace.
NON-UNIFORM ORDERED: ALL BUT

\[ \langle |e^{im_1}|^2 \rangle \text{ ZERO, } \langle |e^{im_1}|^2 \rangle \neq 0. \]

LAST: ANALOGUE OF GUPSE-SONDHI STRIPED PHASE.

\[ S^2 \text{ has Dirac operator with no fermion doubling, instantons,} \]

\[ \text{SEEMS} \text{ BETTER THAN LATTICE FOR SYMMETRIES} \]

SUSY: REPLACE SU(2) BY OSP(2,1) \Rightarrow Fuzzy Sphere becomes fuzzy \text{ N=1 SUSY SUPERSPHERE. SIMULATIONS STARTING.} \]

STRINGS: MYERS: IF N D-BRANES ARE CLOSE, TRANSVERSE COORDINATES \( \Phi \) BECOME N X N MATRICES. ACTION:
$S = \text{Tr} \left[ \phi_i \phi_j \right] + \text{Tr} \left[ \phi_i \phi_j \right] + \text{Tr} \phi_{ijk} \phi_{ijk}$

$\phi_{ijk}$ totally antisymmetric.

$\delta S = 0 \Rightarrow \left[ \phi_i, \phi_j \right] - c \phi_{ijk} \phi_k$ give solutions if $\phi_{ijk}$ structure constants of simple compact Lie group.

Thus possibilities:

- $\phi_i = e L_i$, $\phi_{ijk} = e \epsilon_{ijk}$, $e = \text{constant}$.

We can have

$L_i$ irreducible, $L_i = l (l+1)$, $2l+1 = n$.

Write $L_i = \mathcal{L}(l)$. One fuzzy sphere.

$L_i = \bigoplus \mathcal{L}(l)_k$: many fuzzy spheres.

$\sum (2l_k+1) \cdot n$. 
II

STABILITY ANALYSIS OF THESE SOLUTIONS AND STUDIES OF THEIR PHASES, INCLUDING NUMERICALLY, BY MANY GROUPS. (TIFR GROUP)

GROENEWOLD-MOYAL PLANE
It is an alternative to lattice methods.

Item 2 is more a probe of Planck-scale physics.

This introductory talk will focus on items 1, 2 as they will dominate this meeting.

Quantum gravity & Spacetime noncommutativity: Heuristics.

(Doplicher, Fredenhagen, Roberts)

1) Space-space noncommutativity:

Compton wavelength $\frac{\hbar}{mc}$ to probe Planck scale $L$ requires $\frac{\hbar}{mc} \lesssim L$ or $m \gtrsim \frac{\hbar}{Lc}$ & Planck mass. Such high mass in this small volume will strongly change gravity.
can become black hole. suggests a fundamental length limiting spatial localisation.

2) similar arguments can be made about time-localisation \( \Rightarrow \) observation of very short time scales require very high energies. They can produce black holes. Black hole horizon will then limit spatial resolution suggesting \( \Delta \bar{x}^2 \geq L^2 \), \( L \): a fundamental length.

Groenewold - the Smoyal plane

\( x_\mu \times x_\nu - x_\nu \times x_\mu = i \theta_{\mu\nu} \), \( x_\mu \): coordinate functions, models above spacetime uncertainties.
IN MID-90'S, Doplicher, Fredenhagen, and Roberts systematically developed \textbf{unitary} QFT's on Groenewold-Moyal plane even with time-space noncommutativity.

Emerges also from string physics & quantum Hall effect.

Groenewold-Moyal plane \( A_\theta (\mathbb{R}^{d+1}) \)

Consists of functions \( \alpha, \beta, \ldots \) on \( \mathbb{R}^{d+1} \) with \( \ast \)-product

\[ \alpha \ast \beta = \alpha e^{-\nu^2 \frac{1}{2} \hat{\sigma} \ast \hat{\sigma}} \beta, \quad \hat{\sigma} \ast \hat{\sigma} = \delta_{\mu}^{\nu} \Theta_{\mu\nu} \]\n
Implies for spacetime coordinates:

\[ [x^\mu, x^\nu]_\ast = -\nu \Theta_{\mu\nu}. \]

\[ x^\mu \ast x^\nu - x^\nu \ast x^\mu. \]
HOW THIS EMERGES FROM STRING PHYSICS

WE SAW:

- **HEISENBERG**'S ARGUMENTS FOR MOYAL PLANE FOR REGULATING QFT.
- DOPPLICHER ET AL. ARGUMENTS FROM QUANTUM GRAVITY.

A 3rd source is quantum gravity Hall effect, and a similar effect in string theory. Thus:

**THE (LANDAU PROBLEM).**

**Electron in 1-2 plane, \( \vec{B} = (0,0,B) \)**

\[ A_\alpha = -\frac{B}{2} E_{\alpha \beta} x^\beta \quad \alpha, \beta = 1,2. \]

\[ L = \frac{m}{2} \dot{x}_\alpha^2 - e \dot{x}_\alpha A_\alpha \]

If \( eB \to \infty \),

\[ L \approx + \frac{eB}{2} (\dot{x}_1 x_2 - \dot{x}_2 x_1) \]
\[
\left[ \hat{x}_a, \hat{x}_b \right] = + \frac{1}{eB} \varepsilon_{ab} \text{ on quantisation}
\]

This result, for \( B \to \infty \), is exact. Then only lowest Landau level populated. If \( P \) projector to this level and \( \hat{x}_a \) actual position operator,

\[
\hat{x}_a = P \hat{x}_a P.
\]

**Strings**

Consider open boson strings ending on Dp-branes.

If \( B_{ij} = -B_{ji} = \text{constant} \) 2-form (Neveu-Schwarz) field

\[
S = \frac{1}{4 \pi \alpha'} \int \left[ \sum [\varepsilon_{ij} \partial_a x^i \partial_a x^j - 2 \pi \alpha' \varepsilon_{ij} \partial_a x^i \cdot \partial_b x^j \varepsilon^{ab} + \text{spinor terms}] \right] \, d\sigma \, dt
\]
As \( d' \to \infty \), \( S_\Sigma = -\frac{2\pi e}{4\pi} \int_\Sigma \theta_{ij} dx^i \wedge dx^j \)

\[
= -\frac{1}{2} \left[ \int_{\Sigma_0} - \int_{\Sigma_1} \right] e \theta_{ij} x^i \frac{dx^j}{dt}
\]

\[ \Rightarrow e \left[ B_{ij}, \hat{x}^j \right] = \text{or} \]

\[ \left[ \hat{x}^j, \hat{x}^k \right] = \frac{1}{e} (B^{-1})_{jk} \Rightarrow \text{THE "GB" PLANE.} \]
Much work on (2) QFT's and renormalisation theory.

(3) Phenomenology: Lorentz group violation, CP, T, CPT, ...

\[ \theta_{\mu\nu} \text{ in } [x_\mu, x_\nu]_x = i \theta_{\mu\nu} \]

But in ~04,05, Chaichian et al., Aschieri et al. popularise Drinfel'd twist which restores full diffeomorphism invariance despite the presence of constant \( \theta_{\mu\nu} \) in \( [x_\mu, x_\nu]_x = i \theta_{\mu\nu} \).

This twist also twists \( \xi \) statistics (Bal et al.).

Much of this known for some time to Majid, Oeckl, Fiore,
DRIN'FELD TWIST TWISTS BOTH

1) ACTION OF DIFFEOS AND

2) STATISTICS.

This brings into question much of prehistory-analysis.

Example: 1) Lorentz invariance need not be violated even if $\Theta_{\mu\nu} \neq 0$. 2) There need be no ultraviolet-infrared (UV-IR) mixing (see later) in absence of gauge fields.

There is also a striking, clean separation of matter from gauge fields due to DRIN'FELD TWIST. Reminiscent of separation of particles and waves in classical theory. † They have to be treated differently (Bales 20)
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There is much more to talk about: Standard Model, gravity, phenomenology, ... much room for fantasy.

Exercise imagination

Not all the names

Seckin Kurkcuoglu - Ankara
Babar Qureshi - Lahore
Sajjan Vaidya
Kumar Gupta
T. R. Govindarajan

A. Pinzul - Moscow
Gianpiero Manganaro - Napoli.