Bosonization of a Finite Number of Non-Relativistic Fermions and Applications

Avinash Dhar
Tata Institute of Fundamental Research, Mumbai

12th Regional Conference on Mathematical Physics
National Center for Physics, Islamabad
March 31, 2006
Contents

- Introduction and Motivation
- Exact Bosonization
- Applications
  - AdS/CFT - Half-BPS States and LLM Geometries
  - Free nonrelativistic fermions on a circle - Tomonaga’s Problem
- Summary
Introduction and Motivation

The idea of bosonization - finding a bosonic system equivalent to a given fermionic system - is almost as old as quantum mechanics itself.
Introduction and Motivation

The idea of bosonization - finding a bosonic system equivalent to a given fermionic system - is almost as old as quantum mechanics itself.

- **Bloch** - earliest observation for the existence of quantized collective bose excitations - sound waves - in a gas of fermions in 3-dimensions
- **Bohm and Pines** - charge density waves - plasma oscillations - in a gas of electrons.
Introduction and Motivation

The idea of bosonization - finding a bosonic system equivalent to a given fermionic system - is almost as old as quantum mechanics itself.

- Bloch - earliest observation for the existence of quantized collective bose excitations - sound waves - in a gas of fermions in 3-dimensions
- Bohm and Pines - charge density waves - plasma oscillations - in a gas of electrons.
- Tomonaga - first important breakthrough in treating a large system of interacting fermions. In a rigorously defined simple one-dimensional model, he showed that interactions between fermions can mediate new collective bosonic d.o.f
Introduction and Motivation

- Non-relativistic fermions have a quadratic dispersion relation - Tomonaga’s treatment is valid only in the low-energy approximation.

- Luttinger later used a strictly linear dispersion relation. Other work - Mattis and Lieb, Haldane, .... => relativistic bosonization due to Coleman and Mandlestam.

- Tomonaga-Luttinger liquid provides an important paradigm in condensed matter physics.
Introduction and Motivation

Non-relativistic fermions in 1-dimension appear in many situations in string theory and quantum field theory.
Introduction and Motivation

- Non-relativistic fermions in 1-dimension appear in many situations in string theory and quantum field theory.

- Non-critical string theory in 2-dimensions
Introduction and Motivation

- Non-relativistic fermions in 1-dimension appear in many situations in string theory and quantum field theory.

- Non-critical string theory in 2-dimensions

- Half-BPS sector of $\mathcal{N} = 4$ super Yang-Mills theory in 4-dimensions
Introduction and Motivation

- Non-relativistic fermions in 1-dimension appear in many situations in string theory and quantum field theory.

- Non-critical string theory in 2-dimensions
- Half-BPS sector of $\mathcal{N} = 4$ super Yang-Mills theory in 4-dimensions
- Yang-Mills theory on a cylinder in 2-dimensions
Introduction and Motivation

- Non-relativistic fermions in 1-dimension appear in many situations in string theory and quantum field theory.

- Non-critical string theory in 2-dimensions
- Half-BPS sector of $\mathcal{N} = 4$ super Yang-Mills theory in 4-dimensions
- Yang-Mills theory on a cylinder in 2-dimensions

This is closely related to Tomonaga’s problem
Introduction and Motivation

A common feature of all these examples is that the fermionic system arises from an underlying matrix quantum mechanics problem

\[ S = \int dt \left\{ \frac{1}{2} \dot{M}^2 - V(M) \right\} \]
A common feature of all these examples is that the fermionic system arises from an underlying matrix quantum mechanics problem

\[ S = \int dt \left\{ \frac{1}{2} \dot{M}^2 - V(M) \right\} \]

\( M \) is an \( N \times N \) matrix
Introduction and Motivation

- A common feature of all these examples is that the fermionic system arises from an underlying matrix quantum mechanics problem

\[ S = \int dt \left\{ \frac{1}{2} \dot{M}^2 - V(M) \right\} \]

- In the \( U(N) \) invariant sector, the matrix model is equivalent to a system of \( N \) non-relativistic fermions \(^a\)

- Jevicki and Sakita \(^b\) used this equivalence to develop a bosonization in the large-\( N \) limit - collective field theory


\(^b\) Nucl. Phys. B165, 511, 1980
Introduction and Motivation

- Bosonization in terms of Wigner phase space density

\[
    u(p, q, t) = \int dx \ e^{-ipx} \sum_{i=1}^{N} \psi_{i}^{\dagger}(q - x/2, t)\psi_{i}(q + x/2, t)
\]

- \( u(p, q, t) \) satisfies two constraints:
  - \( \int \frac{dpdq}{2\pi} u(p, q, t) = N \)
  - \( u \ast u = u \)

\(^{a}\text{Dhar, Mandal and Wadia, hep-th/9204028; 9207011; 9309028}\)
Introduction and Motivation

- Bosonization in terms of Wigner phase space density

$$u(p, q, t) = \int dx \ e^{-ipx} \sum_{i=1}^{N} \psi_i^\dagger(q - x/2, t) \psi_i(q + x/2, t)$$

- $u(p, q, t)$ satisfies two constraints:
  - $\int \frac{dp dq}{2\pi} u(p, q, t) = N$
  - $u \ast u = u$

Many more variables than are necessary

---

\textsuperscript{a}Dhar, Mandal and Wadia, hep-th/9204028; 9207011; 9309028.
The Setup:

- each can occupy a state in an infinite-dimensional Hilbert space $\mathcal{H}_f$

- there is a countable basis of $\mathcal{H}_f$: $\{|m\rangle, m = 0, 1, \ldots, \infty\}$

- creation and annihilation operators $\psi_m^\dagger, \psi_m$ create and destroy particles in the state $|m\rangle$, $\{\psi_m, \psi_n^\dagger\} = \delta_{mn}$

- total number of fermions is fixed:

$$\sum_n \psi_n^\dagger \psi_n = N$$
Exact Bosonization

The $N$-fermion states are given by (linear combinations of)

$$|f_1, \cdots, f_N\rangle = \psi_{f_N}^\dagger \cdots \psi_{f_2}^\dagger \psi_{f_1}^\dagger |0\rangle_F,$$

- $|0\rangle_F$ is Fock vacuum
- $f_k$ are ordered $0 \leq f_1 < f_2 < \cdots < f_N$

Repeated applications of the bilinear $\psi_m^\dagger \psi_n$ gives any desired state
Exact Bosonization

\[ f_1 \]
\[ f_2 \]
\[ f_3 \]
\[ \cdots \]
\[ f_{N-2} \]
\[ f_{N-1} \]
\[ f_N \]
Exact Bosonization

Bosonization: 

- Introduce the bosonic operators

\[ \sigma_k, \ k = 1, 2, \cdots, N \]

- and their conjugates

\[ \sigma_k^\dagger, \ k = 1, 2, \cdots, N \]

\(^a\text{Dhar, Mandal and Suryanarayana, hep-th/0509164}\)
Exact Bosonization
Exact Bosonization

\[ \sigma_k \]

\[ f_N, f_{N-1}, f_{N-k+2}, f_{N-k+1} \]

\[ -f_N, -f_{N-1}, -f_{N-k+2}, -f_{N-k+1} \]
Exact Bosonization

By definition:

\[ \sigma_k \sigma_k^\dagger = 1 \]

\[ \sigma_k^\dagger \sigma_k = 1, \text{ if } \sigma_k \text{ does not annihilate the state} \]
By definition:

\[ \sigma_k \sigma_k^\dagger = 1 \]

\[ \sigma_k^\dagger \sigma_k = 1, \text{ if } \sigma_k \text{ does not annihilate the state} \]
Exact Bosonization

By definition:

- \( \sigma_k \sigma_k^\dagger = 1 \)
- \( \sigma_k^\dagger \sigma_k = 1 \), if \( \sigma_k \) does not annihilate the state

For \( k \neq l \), \( [\sigma_k, \sigma_l^\dagger] = 0 \)
Exact Bosonization

- Introduce creation (annihilation) operators $a_k^\dagger$ ($a_k$) which satisfy the standard commutation relations

  $$[a_k, a_l^\dagger] = \delta_{kl}, \quad k, l = 1, \cdots, N$$

- The states of the bosonic system are given by (a linear combination of)

  $$|r_1, \cdots, r_N\rangle = \frac{(a_1^\dagger)^{r_1} \cdots (a_N^\dagger)^{r_N}}{\sqrt{r_1! \cdots r_N!}} |0\rangle$$
Exact Bosonization

Now, make the following identifications

\[ \sigma_k = \frac{1}{\sqrt{a_k^\dagger a_k + 1}} a_k; \quad \sigma_k^\dagger = a_k^\dagger \frac{1}{\sqrt{a_k^\dagger a_k + 1}} \]

together with the map

\[ r_N = f_1; \quad r_k = f_{N-k+1} - f_{N-k} - 1, \quad k = 1, 2, \ldots N - 1 \]

For the Fermi vacuum, \( f_{k+1} = f_k + 1 \) and so \( r_k = 0 \) for all \( k \) => Fermi vacuum = Bose vacuum

---

\(^a\)Suryanarayana, hep-th/0411145
Exact Bosonization

- The $\sigma_k, k = 1, 2, \ldots, N$ are necessary and sufficient
- Any bilinear $\psi_n^\dagger \psi_m$ can be built out of $\sigma_k$'s
The $\sigma_k, k = 1, 2, \cdots, N$ are necessary and sufficient.

Any bilinear $\psi_n^\dagger \psi_m$ can be built out of $\sigma_k$'s.
The $\sigma_k$, $k = 1, 2, \cdots, N$ are necessary and sufficient

- Any bilinear $\psi_n^\dagger \psi_m$ can be built out of $\sigma_k$'s
Exact Bosonization

Generic properties of the bosonized theory:

- Each boson can occupy only a finite number of different states, as a consequence of a finite number of fermions => a cut-off or graininess in the bosonized theory!
Exact Bosonization

Generic properties of the bosonized theory:

- Each boson can occupy only a finite number of different states, as a consequence of a finite number of fermions => a cut-off or graininess in the bosonized theory!

- There is no natural “space” in the bosonic theory - in the examples we will discuss, a spatial direction will emerge in the low-energy large-\(N\) limit.
Exact Bosonization

Generic properties of the bosonized theory:

- Each boson can occupy only a finite number of different states, as a consequence of a finite number of fermions => a cut-off or graininess in the bosonized theory!

- There is no natural “space” in the bosonic theory - in the examples we will discuss, a spatial direction will emerge in the low-energy large-$N$ limit.

- In applications involving matrix quantum mechanics, our bosonization can be considered to be an exact solution of the matrix problem in the singlet sector
**Exact Bosonization**

- The non-interacting fermionic Hamiltonian:

\[
H = \sum_n \mathcal{E}(n) \psi_n^\dagger \psi_n
\]
Exact Bosonization

- The non-interacting fermionic Hamiltonian:

\[ H = \sum_n \mathcal{E}(n) \psi_n^\dagger \psi_n \]

- The bosonized Hamiltonian:

\[ H = \sum_{k=1}^{N} \mathcal{E}(\hat{n}_k), \quad \hat{n}_k = \sum_{i=k}^{N} a_i^\dagger a_i + N - k \]
Exact Bosonization

- The non-interacting fermionic Hamiltonian:

\[ H = \sum_n \mathcal{E}(n) \psi_n^\dagger \psi_n \]

- The bosonized Hamiltonian:

\[ H = \sum_{k=1}^N \mathcal{E}(\hat{n}_k), \quad \hat{n}_k = \sum_{i=k}^N a_i^\dagger a_i + N - k \]

- What about fermion interactions? These can also be included since the generic bilinear \( \psi_n^\dagger \psi_m \) has a bosonized expression
Half-BPS states and LLM geometries

- Space-time, gravity lagrangians and gravitons - low-energy emergent properties of an underlying microscopic dynamics
- String theory is a consistent theory of quantum gravity => we should be able to test these ideas
- AdS/CFT correspondence => a precise setting in which to explore these ideas
Half-BPS states and LLM geometries

- Space-time, gravity lagrangians and gravitons - low-energy emergent properties of an underlying microscopic dynamics
- String theory is a consistent theory of quantum gravity => we should be able to test these ideas
- AdS/CFT correspondence => a precise setting in which to explore these ideas

- Classic example of dual pair - $\mathcal{N} = 4$ SYM and string theory on $\text{AdS}_5 \times S^5$. No hint of 10-d space-time or gravitons in the SYM theory!
- This duality => weakly coupled low-energy type IIB gravity on $\text{AdS}_5 \times S^5$ and strongly coupled $\mathcal{N} = 4$ SYM theory in the large-N limit have exactly the same physical content
Half-BPS states and LLM geometries

- LLM work - a small new window of opportunity.

- Limited to half-BPS sector, but hopefully has some wider lessons

- SYM - half-BPS states are described by a holomorphic sector of quantum mechanics of an $N \times N$ complex matrix $Z$ in a harmonic potential

- This system can be shown to be equivalent to the quantum mechanics of an $N \times N$ hermitian matrix $Z$ in a harmonic potential

---

\(^{a}\) Lin, Lunin and Maldacena, hep-th/0409174

\(^{b}\) Takayama and Tsuchiya, hep-th/0507070
Half-BPS states and LLM geometries

- Gauge invariance $\Rightarrow$ physical observables on boundary are $U(N)$-invariant traces:
  \[ \text{tr} Z^k, \quad k = 1, 2, \ldots, N \]

- Physical states $\Leftrightarrow$ operators
  \[ (\text{tr} Z^{k_1})^{l_1} (\text{tr} Z^{k_2})^{l_2} \ldots \]

- Total number of $Z$'s is a conserved RR charge
  \[ Q = \sum_i k_i l_i. \text{ BPS condition} \Rightarrow E = Q \]
At large $N$ there is a semiclassical picture of the states of this system in terms of droplets of fermi fluid in phase space.
At large $N$ there is a semiclassical picture of the states of this system in terms of droplets of fermi fluid in phase space.

ground state distribution
Half-BPS states and LLM geometries

At large $N$ there is a semiclassical picture of the states of this system in terms of droplets of fermi fluid in phase space

small fluctuations around the ground state
Half-BPS states and LLM geometries

- By explicitly solving equations of type IIB gravity, LLM showed that there is a similar structure in the classical geometries in the half-BPS sector!

- LLM solutions - two of the space coordinates are identified with the phase space of a single fermion => noncommutativity in two space directions in the semiclassical description.

- Small fluctuations around AdS space, i.e low-energy graviton excitations \( \equiv \) low-energy fluctuations of the fermi vacuum.

---

\( ^a \) Mandal, hep-th/0502104
\( ^b \) Grant, Maoz, Marsano, Papadodimas and Rychkov, hep-th/0505079
\( ^c \) Maoz and Rychkov, hep-th/0508059
\( ^d \) Dhar, hep-th/0505084
Half-BPS states and LLM geometries

- Motivation for our work - on the CFT side the half-BPS system can be quantized exactly in terms of our bosons => window of opportunity to learn about aspects of quantum gravity.

- At finite $N$, only the low-energy excitations on the boundary can be identified with low-energy ($<< N$) gravitons in the bulk.

- The single-particle graviton excitations are related to our bosons. On the boundary, these states are:

$$\beta_m^\dagger |0\rangle = \sum_{n=1}^{m} (-1)^{n-1} \sqrt{\frac{(N + m - n)!}{2^m (N - n)!}} \sigma_1^{m-n} \sigma_n^\dagger |0\rangle$$

---

$^a$Dhar, Mandal and Smedback, hep-th/0512312
One can compute exactly the correlation functions

\[ \langle \beta_{k_1}^\dagger \beta_{k_2}^\dagger \cdots \rangle \]
Half-BPS states and LLM geometries

- One can compute exactly the correlation functions
  \[ \langle \beta_{k_1}^\dagger \beta_{k_2}^\dagger \cdots \rangle \]

- On the boundary:
Half-BPS states and LLM geometries

- One can compute exactly the correlation functions
  \[ < \beta^\dagger_{k_1} \beta^\dagger_{k_2} \cdots > \]

- On the boundary:
  - at low energies, perturbation theory is good and reproduces supergravity answers; there is an effective cubic hamiltonian
Half-BPS states and LLM geometries

- One can compute exactly the correlation functions
  \[ < \beta^\text{\dagger}_{k_1} \beta^\text{\dagger}_{k_2} \cdots > \]

- On the boundary:
  - at low energies, perturbation theory is good and reproduces supergravity answers; there is an effective cubic hamiltonian
  - perturbation theory breaks down for \( \beta \)'s with energy of order \( \sqrt{N} \)
Half-BPS states and LLM geometries

- One can compute exactly the correlation functions
  \[ < \beta_{k_1}^{\dagger} \beta_{k_2}^{\dagger} \cdots > \]

- On the boundary:
  - at low energies, perturbation theory is good and reproduces supergravity answers; there is an effective cubic hamiltonian
  - perturbation theory breaks down for \( \beta \)'s with energy of order \( \sqrt{N} \)
  - At energies of order \( N \), the \( \beta \) interactions grow exponentially with \( N \)
Half-BPS states and LLM geometries

- One can compute **exactly** the correlation functions
  \[ < \beta_{k_1}^\dagger \beta_{k_2}^\dagger \cdots > \]

- In the bulk:
Half-BPS states and LLM geometries

- One can compute exactly the correlation functions
  \[ < \beta_{k_1}^{\dagger} \beta_{k_2}^{\dagger} \cdots > \]

- In the bulk:
  - gravitons with energies larger than \( \sqrt{N} \) have a size smaller than 10-dim Planck scale
Half-BPS states and LLM geometries

- One can compute exactly the correlation functions
  \[ \langle \beta_{k_1}^\dagger \beta_{k_2}^\dagger \cdots \rangle \]

- In the bulk:
  - gravitons with energies larger than \( \sqrt{N} \) have a size smaller than 10-dim planck scale
  - nonlocal solitonic excitations with energy of order \( N \)
    - giant gravitons \(^a\)

---

\(^a\)McGreevy, Susskind and Toumbas, hep-th/0003075
Half-BPS states and LLM geometries

- One can compute exactly the correlation functions
  \[ \langle \beta_{k_1}^\dagger \beta_{k_2}^\dagger \cdots \rangle \]

- In the bulk:
  - gravitons with energies larger than \( \sqrt{N} \) have a size smaller than 10-dim planck scale
  - nonlocal solitonic excitations with energy of order \( N \)
    - giant gravitons
  - The size of giant gravitons is larger than 10-dim planck scale for energies larger than \( \sqrt{N} \)

---

\( ^a \)McGreevy, Susskind and Toumbas, hep-th/0003075
Half-BPS states and LLM geometries

On the boundary, single-particle giant graviton states map to linear combinations of multi-graviton states. Example:

\[ |\text{giant graviton of energy 2}\rangle = (\beta_1^\dagger)^2 - \beta_2^\dagger |0\rangle \]

\(^a\)Balasubramanian, Berkooz, Naqvi and Strassler, hep-th/0107119
Half-BPS states and LLM geometries

On the boundary, single-particle giant graviton states map to linear combinations of multi-graviton states. Example:

\[ |\text{giant graviton of energy 2}\rangle = (\beta_1^+)^2 - \beta_2^+ |0\rangle = a_2^+ |0\rangle \]

\(^a\)Balasubramanian, Berkooz, Naqvi and Strassler, hep-th/0107119
Boundary states corresponding to single-particle bulk giant states are our single-particle bosonic states:

$$|\text{giant graviton of energy } k\rangle = a^\dagger_k |0\rangle$$
Half-BPS states and LLM geometries

- Boundary states corresponding to single-particle bulk giant states are our single-particle bosonic states:

  \[ |\text{giant graviton of energy } k \rangle = a_k^\dagger |0\rangle \]

- Hamiltonian:

  \[ H_F = \sum_n \mathcal{E}(n) \psi_n^\dagger \psi_n \Rightarrow H_B = \sum_{k=1}^N k a_k^\dagger a_k \]
Half-BPS states and LLM geometries

- Boundary states corresponding to single-particle bulk giant states are our single-particle bosonic states:

\[ |\text{giant graviton of energy } k\rangle = a_k^\dagger |0\rangle \]

- Hamiltonian: \( H_F = \sum_n \mathcal{E}(n) \psi_n^\dagger \psi_n \Rightarrow H_B = \sum_{k=1}^N k a_k^\dagger a_k \)

- Discrete space?

\[ \phi(\theta_n) = \sum_{k=1}^N \left( e^{ik\theta_n} a_k + e^{-ik\theta_n} a_k^\dagger \right), \quad \theta_n = \frac{2\pi n}{N} \]
Half-BPS states and LLM geometries

Summary (half-BPS sector):

- Low-energy fluctuations of the metric around AdS are adequately described by gravitons.

- At moderately high energies of order $\sqrt{N}$, perturbative gravity breaks down; one must now sum to all orders in $1/N$ to get correct answers.

- At very high energies of order $N$, gravitons cease to provide a meaningful description; instead we must now use a new set of d.o.f., namely the giant gravitons, which are weakly coupled at high energies.
Free fermions on a circle

We will mainly discuss the free fermion problem. Interactions can be taken into account once the free part has been dealt with properly.

The free hamiltonian:

\[ H = -\frac{\hbar^2}{2m} \int_0^L dx \, \chi^\dagger(x) \partial_x^2 \chi(x) \]

\(^a\text{Dhar and Mandal, hep-th/0603154}\)
Free fermions on a circle

- We will mainly discuss the free fermion problem. Interactions can be taken into account once the free part has been dealt with properly.

- Hamiltonian in terms of fourier modes:

\[ H = \omega \hbar \sum_{n=-\infty}^{\infty} n^2 \chi_n^\dagger \chi_n, \quad \omega \equiv \frac{2\pi^2 \hbar}{mL^2} \]

---

\[a\] Dhar and Mandal, hep-th/0603154
Free fermions on a circle

We will mainly discuss the free fermion problem. Interactions can be taken into account once the free part has been dealt with properly.

Hamiltonian in terms of fourier modes:

$$H = \omega \hbar \sum_{n=-\infty}^{\infty} n^2 \chi_n^\dagger \chi_n, \quad \omega \equiv \frac{2\pi^2 \hbar}{mL^2}$$

To apply our bosonization rules, need to introduce an ordering in the spectrum. For example, replace

$$n^2 \rightarrow (n + \epsilon)^2$$

---

\textsuperscript{a}Dhar and Mandal, hep-th/0603154
Free fermions on a circle

Diagram showing the arrangement of fermions on a circle, with indices from 0 to 10 and signs indicating positive or negative fermions.
Effectively, we have set $\chi_{+n} = \psi_{2n}$ and $\chi_{-n} = \psi_{2n-1}$.
Free fermions on a circle

- Effectively, we have set $\chi^+_n = \psi_{2n}$ and $\chi^-_n = \psi_{2n-1}$
- Fermionic hamiltonian:

$$H = \omega \hbar \sum_{n=1}^{\infty} \left( \frac{n + e(n)}{2} \right)^2 \psi_n^\dagger \psi_n$$
Free fermions on a circle

- Effectively, we have set $\chi_+ n = \psi_{2n}$ and $\chi_- n = \psi_{2n-1}$

- Fermionic hamiltonian:

$$H = \omega \hbar \sum_{n=1}^{\infty} \left( \frac{n + e(n)}{2} \right)^2 \psi_n^\dagger \psi_n$$

- Bosonized hamiltonian:

$$H = \omega \hbar \sum_{k=1}^{N} \left( \frac{\hat{n}_k + e(\hat{n}_k)}{2} \right)^2$$

where $\hat{n}_k = \sum_{i=k}^{N} a_i^\dagger a_i + N - k$
Free fermions on a circle

Large-$N$ low energy limit: $H = H_F + H_0 + H_1$
Free fermions on a circle

- Large-\(N\) low energy limit: \(H = H_F + H_0 + H_1\)

\[
H_0 = \frac{\hbar \omega N}{2} \left( \sum_{k=1}^{N} k a_k^{\dagger} a_k + \hat{\nu} \right)
\]

- \(\hat{\nu} = N_- - N_{-F} = \sum_{k=1}^{N} (e(\hat{n}_k) - e(N - k))\) is the number of excess fermions in negative momentum states over and above the number in fermi vacuum

- \(H_1\) is order one on excited states whose energy is low compared to \(N\)
Free fermions on a circle

- The massless collective boson:
Free fermions on a circle

- The massless collective boson:

- The partition function

\[ Z_N = \sum e^{-\beta H_0} \]
Free fermions on a circle

- The massless collective boson:

- The partition function

\[ Z_N = \sum e^{-\beta H_0} \]

- In the limit \( N \to \infty \), the partition function turns out to be

\[ Z_\infty = \sum_{\nu=\pm \infty} q^{\nu^2} \left[ \prod_{n=1}^{\infty} (1 - q^n)^{-1} \right]^2; \quad q = e^{-\hbar \omega N \beta} \]
Free fermions on a circle

States ($\nu = 0$):
Free fermions on a circle

States ($\nu = 0$):

- states at level $l$ have energy $E_{0l} = \hbar \omega N l$ - these include multiparticle states of both chiralities
Free fermions on a circle

States \((\nu = 0)\):

- states at level \(l\) have energy \(E_{0l} = \hbar \omega N l\) - these include multiparticle states of both chiralities

- example, \(l = 2\):

\[
(\sigma_1^{\dagger})^4|0\rangle, \sigma_1^{\dagger}\sigma_3^{\dagger}|0\rangle, (\sigma_1^{\dagger})^2\sigma_2^{\dagger}|0\rangle, \sigma_4^{\dagger}|0\rangle, (\sigma_2^{\dagger})^2|0\rangle
\]

- first two have momentum of opposite sign to the next two; the last state has zero momentum

- first four states: two single-particle and two 2-particle states of each chirality; the last state is non-chiral 2-particle state
Free fermions on a circle

$1/N$ corrections:
Free fermions on a circle

1/N corrections:

linear combinations exist in which expectation value of $H_1$ vanishes:

$$\frac{1}{\sqrt{2}}[\sigma_1^4 \pm \sigma_1^4 \sigma_3^4]|0\rangle; \quad \frac{1}{\sqrt{2}}[(\sigma_1^4)^2 \sigma_2^4 \pm \sigma_4^4]|0\rangle$$
Free fermions on a circle

1/N corrections:

linear combinations exist in which expectation value of $H_1$ vanishes:

$$\frac{1}{\sqrt{2}}[(\sigma_1^\dagger)^4 \pm \sigma_1^\dagger \sigma_3^\dagger]|0\rangle; \quad \frac{1}{\sqrt{2}}[(\sigma_1^\dagger)^2 \sigma_2^\dagger \pm \sigma_4^\dagger]|0\rangle$$

such linear combinations exist at all levels; at each level there is one linear combination which is identical to an appropriate mode of the fermion density!

$$\frac{1}{\sqrt{l}} \sum_n \psi_n^\dagger \psi_{n+2l}|F_0\rangle \equiv \rho_l^\dagger |0\rangle = \frac{1}{\sqrt{l}} \sum_k^{[2l]} (\sigma_1^\dagger)^{2l-k} \sigma_k^\dagger |0\rangle$$
Free fermions on a circle

- Cubic interaction:
Free fermions on a circle

Cubic interaction:

\[ H_1 \rho_l^\dagger |0\rangle = \hbar \omega \sum_{m=1}^{l-1} c_m^l \rho_m^\dagger \rho_{(l-m)}^\dagger |0\rangle, \quad l \leq \frac{N+1}{2} \]
Free fermions on a circle

- Cubic interaction:

\[ H_1 \rho_l \dagger |0\rangle = \hbar \omega \sum_{m=1}^{l-1} c_m^l \rho_m \rho_{(l-m)} \dagger |0\rangle, \quad l \leq \frac{N + 1}{2} \]

- The coefficient:

\[ c_m^l = \sqrt{lm(l - m)} \]
Free fermions on a circle

Beyond low-energy perturbation theory:
Free fermions on a circle

Beyond low-energy perturbation theory:

states created by modes of fermion density have an extra term at high energies:

\[
\hat{\rho}_l^\dagger |0\rangle = \rho_l^\dagger |0\rangle + \frac{1}{\sqrt{l}} \sum_{n=1}^{l} \psi_{2(l-n)}^\dagger \psi_{2n-1} |0\rangle
\]
Free fermions on a circle

Beyond low-energy perturbation theory:

- states created by modes of fermion density have an extra term at high energies:

\[
\tilde{\rho}_l^\dagger |0\rangle = \rho_l^\dagger |0\rangle + \frac{1}{\sqrt{l}} \sum_{n=1}^{l} \psi_{2(l-n)}^\dagger \psi_{2n-1} |0\rangle
\]

- last term cannot be ignored at high energies; it is in fact a \( \nu = -1 \) state!
Free fermions on a circle

\[
\tilde{\rho}^\dagger \tilde{\rho}^\dagger (\frac{N+3}{2} - m) |0\rangle = \rho^\dagger \rho^\dagger (\frac{N+3}{2} - m) |0\rangle + \frac{1}{\sqrt{m(\frac{N+3}{2} - m)}} \sigma_1^\dagger \sigma_{N-1}^\dagger |0\rangle
\]

last term will contribute in

\[
\langle 0 | \tilde{\rho}^\dagger + \frac{N+3}{2} \tilde{\rho} + m \tilde{\rho}^\dagger + (\frac{N+3}{2} - m) |0\rangle
\]
Free fermions on a circle

Exact partition function for finite $N$ ($H_0$ part only):

$$Z_N = \sum_{\nu = -\frac{N-1}{2}}^{\frac{N+1}{2}} q^{\nu^2} \prod_{n=1}^{\frac{N+1}{2}} \frac{1}{1 - q^n} \prod_{n=1}^{\frac{N-1}{2}} \frac{1}{1 - q^n}$$
Free fermions on a circle

- Exact partition function for finite $N$ ($H_0$ part only):

$$Z_N = \sum_{\nu = -\frac{N-1}{2}}^{\frac{N+1}{2}} q^{\nu^2} \prod_{n=1}^{\frac{N+1}{2} - \nu} (1 - q^n)^{-1} \prod_{n=1}^{\frac{N-1}{2} + \nu} (1 - q^n)^{-1}$$

- For large-$N$:

$$Z_N = (1 + 2q - q^{\frac{N+1}{2}})(1 - q^{\frac{N+1}{2}}) \left[ \prod_{n=1}^{\infty} (1 - q^n)^{-1} \right]^2$$
Free fermions on a circle

- Exact partition function for finite \( N \) (\( H_0 \) part only):

\[
Z_N = \sum_{\nu=-\frac{N-1}{2}}^{\frac{N+1}{2}} q^{\nu^2} \prod_{n=1}^{N+1-\nu} (1 - q^n)^{-1} \prod_{n=1}^{\frac{N-1}{2}+\nu} (1 - q^n)^{-1}
\]

- For large-\( N \):

\[
Z_N = (1 + 2q - q^{\frac{N+1}{2}})(1 - q^{\frac{N+1}{2}}) \left[ \prod_{n=1}^{\infty} (1 - q^n)^{-1} \right]^2 \sim O(e^{-N})
\]

- Nonperturbative effects in 2-d YM
- Black-hole counting and baby universes
Free fermions on a circle

Summary:

- Tomonaga’s problem has an exact solution in terms of our bosons. Low-energy local cubic collective field theory can be derived; the collective field is a linear combination of multi-particle states of our bosons, like the graviton in the LLM case.

- Our bosonization goes beyond this low-energy local limit, but then there is no natural local space-time field theory interpretation.

- Density-density interactions, as in a system of electrons with Coulomb interactions, can be incorporated - easy at low-energies, but requires more work at high energies.
SUMMARY

- We have developed a simple and exact bosonization of a finite number of non-relativistic fermions; we discussed here applications to concrete problems in different areas of physics.

- Our bosonization trades finiteness of the number of fermions for finite dimensionality of the single-particle boson Hilbert space.

- The bosonized theory is inherently grainy; in the specific applications we discussed, a local space-time field theory emerges only in the large-$N$ and low-energy limit.

- Bosonization of finite number of fermions in higher dimensions?