Uniqueness of the Foliation of Spherically Symmetric Static Spacetimes by Flat Spacelike Hypersurfaces Corresponding to Freely Falling Observers

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Foliation
Foliation by Flat Spacelike Hyperurfaces
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Foliation

- Splitting a space \((S)\) into a sequence of subspaces \((SS)\) such that each and every point of the space lies on one and only one of the subspaces is called a foliation.

- **Codim of Foliation** = \(\dim(S) - \dim(SS)\)

- Subspaces \((SS)\) are called **hypersurfaces** if Codimension = 1
A hypersurface is flat if all the components of the Riemann Curvature Tensor are zero.

Consider Spherically Symmetric Static Spacetime metric

\[ ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} dr^2 - r^2 d\Omega^2 \]

Where

\[ d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \]
To obtain flat hypersurfaces solve

\[ R^i_{\ jkl} = 0 \quad (i, j, k, l = 1, 2, 3) \]

**Instead:** Writing unit tangent vector to the world-line of the observer falling freely from infinity as \( n \) and the the unit tangent vector to the hypersurface as \( T \), we require that

\[
n \cdot n = 1 = -T \cdot T, \quad T \cdot n = 0 \quad ---- \alpha
\]

\[
\frac{dt}{dr} = \pm \frac{e^{\frac{\lambda - \nu}{2}} \sqrt{k^2 - e^{-\lambda}}}{k}
\]

We have flat Hypersurfaces if \( k=1 \).
Now consider hypersurface

\[ f(t, r, \theta, \phi) = 0 \]

Considering spherical symmetry the above hypersurface in explicit form can be given as

\[ t = F(r) \]

The induced metric (of hypersurfaces) is

\[ ds^2 = -\left( e^{\lambda(r)} - e^{\nu(r)} F'^2 \right) dr^2 - r^2 d\Omega^2 \]
For the induced metric to be flat a necessary but not sufficient condition

i.e. Ricci Scalar: \( R = 0 \)

\[
\frac{r \left( -\lambda' e^\lambda + \nu' e^\nu F''^2 + 2 e^\nu F'F'' \right)}{\left( e^\lambda - e^\nu F''^2 \right)^2}
+ \frac{1 - e^\lambda + e^\nu F''^2}{e^\lambda - e^\nu F''^2} = 0 \quad (1)
\]
Using the substitution

\[ k^2(r) = \frac{1}{e^{\lambda(r)} - e^{\nu(r)} F'^2} \]

Equation (1) becomes

\[ 2rkk' + k^2 - 1 = 0, \]

and we have the solution

\[ k^2(r) = 1 - \frac{c}{r} \quad (c \text{ is arbitrary constant}) \]
The induced metric is then
\[ ds^2_3 = -\frac{dr^2}{1 - \frac{c}{r}} - r^2 d\Omega^2 \]

The above metric is flat i.e.
\[ R^1_{212} = R^1_{313} = R^2_{323} = 0 \quad \text{if} \quad c = 0 \]

\[ \therefore k(r) = 1. \]

The Flat Hypersurfaces are then given as:
\[ t = F(r) = \int e^{\frac{\lambda - \nu}{2}} \sqrt{1 - e^{-\lambda}} \, dr. \]
The mean extrinsic curvature, $K$, of these hypersurfaces is

$$K = e^{-\frac{\nu + \lambda}{2}} \left( \frac{\nu e^\nu}{2\sqrt{1-e^\nu}} - \frac{2\sqrt{1-e^\nu}}{r} \right)$$