**Geometry and Symmetry in General Relativity**

Graham Hall (Aberdeen)

1 Introduction

$M$ is a 4-dimensional Hausdorff connected smooth manifold

$g$ is a smooth Lorentz metric on $M$ $(\gamma, t, t, t)$

$(M, g)$ is a space-time.

$\nabla$ is the Levi-Civita connection from $g$ with Christoffel symbols $\Gamma^a_{bc}$

$\text{Riem}$ is curvature tensor from $\nabla$ $(R^a_{bcd})$

$\text{Ricc}$ is Ricci tensor from $\nabla$ $(R_{ab})$

$\triangledown$ covariant derivative, partial derivative

$L$ Lie derivative
2 The Geometry of Symmetry

$U$ is open coordinate domain in $M$ with coordinates $x^a$ and $f$ is a bijective map $f: U \to V (= f(u))$ so $f^{-1}: V \to U$.

The map $f$ is smooth so $f$ is a local diffeomorphism.

So $y^a = x^a \circ f^{-1}$ is a coordinate system on $V$.

Let $T$ be a tensor on $M$.

Define $f$ to be a local symmetry of $T$ if the components of $T$ at $p$ in the system $x^a$ equal the components of $T$ at $f(p)$ in the system $y^a$ for each $p \in U$. ($\Leftrightarrow f^*T = T$)
Now let $X$ be a smooth vector field on $M$. A curve $x^a(t)$ in some coordinate domain of $M$ is an integral curve of $X$ starting from $p \in M$ if $\frac{dx^a}{dt} = X^a$ and if $x^a(0) = p$.

**Theorem** There exists an open neighbourhood $U$ of $p$ and $\varepsilon > 0$ such that for any $q \in U$ there is an integral curve $x^a(t)$ of $X$ which starts from $q$ and which is defined for each $t \in (-\varepsilon, \varepsilon)$.

So, for each $q \in U$ and for each $t \in (-\varepsilon, \varepsilon)$ we can "move" $q$ along the integral curve of $X$, starting from $q$, a (curve) parameter distance $t$ along this curve, to a point, say, $q'$.

Thus, for this $t \in (-\varepsilon, \varepsilon)$ each $q \in U$ is moved a parameter distance $t$ in this way and gives rise to a map $q_t : U \rightarrow V$ ($= q_t(U)$) where $q_t(q) = q'$. These maps, for the choices of $U$ and $t$, are called the local flows or local diffeomorphisms of $X$. 
So let us define a **symmetry of a Tensor** $T$ as a vector field $X$ such that each local flow of $X$ is a local symmetry of $T$.

Then the statement that $X$ is a symmetry of $T$ is equivalent (by definition) to the statement that $\Phi_t^* T = T$ for each local flow $\Phi_t$ of $X$. This is then equivalent to the condition that

$$L_X T = 0$$

(1)

[One could change the symmetry condition $f^* T = T$ to, for example, $f^* T = \lambda T$ for some function $\lambda: U \to \mathbb{R}$, or to some other well-defined geometrical restriction on $f$]

The condition (1) is sometimes more useful than the relations $\Phi_t^* T = T$. 
Perhaps the most important symmetry in general relativity (or in differential geometry) occurs when $T = 0$. Thus we study the equation $\mathcal{L}_X g = 0$ (*Killing's equation*).

An equivalent form of it is

$$X_{a;b} + X_{b;a} = 0 \iff X_{a;b} = F_{ab} (-F_{ba}) \quad (2)$$

Note: if $p \in M$ and $X(p) \neq 0$, choose coordinates $x^a$ so that $X^a = (1, 0, 0, 0) = \delta^a_1$. Then (2) is equivalent to $\frac{\partial g_{ab}}{\partial x^1} = 0$ [and the integral curves of $X$ are of the form $t \mapsto (a + t, b, c, d)$ with $a, b, c, d \in \mathbb{R}$].

The skew-symmetric tensor $F$ is called the *Killing bivector*.

Note that the condition $\phi_t^* g = g$ on the local flows of $X$ is that $\phi_t$ be a *local isometry* of $(M, g)$. 
Other Symmetries are:

(i) When each $\Phi_t$ satisfies $\Phi_t^* g = e^{\sigma} g$ for $\sigma: U \rightarrow \mathbb{R}$ (i.e. $\Phi_t$ a conformal map).

Then $L_X g = \nabla g \psi: M \rightarrow \mathbb{R}$.

and $X$ is a conformal vector field.

(ii) When each $\Phi_t$ preserves (local) geodesics.

Then $X$ is called a projective vector field.

If, in addition, each $\Phi_t$ also preserves affine parameters, the $X$ is called affine.

**Notation**

Let $K(M)$, $H(M)$, $C(M)$, $A(M)$, $P(M)$ denote the Lie algebras of Killing, homothetic, conformal, affine and projective vector fields on $M$. 


3 Symmetry Orbits.

Let \( X_1, \ldots, X_k \in K(M) \). Let their associated local flows be \( \phi_t^1 \cdots \phi_t^k \).

Let \( p \in M \) and (assuming the following are defined) consider the following "movement" of \( p \):

\[
p \rightarrow \phi_{t_1}^1(p) \rightarrow \phi_{t_2}^2(\phi_{t_1}^1(p)) \rightarrow \ldots
\]

All points which can be "reached" in this way from \( p \) make up the orbit through \( p \) and associated with \( K(M) \).

These orbits are submanifolds of \( M \) and integral manifolds of \( K(M) \). That is the tangent space to the orbit through \( p \) is the vector space \( K(M)_p = \{ X(p) : X \in K(M) \} \).

Note The dimension of \( K(M)_p \) may change with \( p \) and so cannot necessarily use Fröbenius theory.
Examples

**Schwarzschild**
or **Reissner–Nordstrom**

**Friedmann–Robertson–Walker–Lemaître**

*timelike orbits*

*spacelike orbits*

**Einstein Static Universe**

*(M is a single orbit - homogeneous case)*

**Plane Waves**

*(non-homogeneous)*
Example.

If $X \in K(M)$ then $L_x R_{ab} = 0$.

Suppose $X(p) = 0$. Then, at $p$,

$$R_{ac} F^c b + R_{bc} F^c a = 0 \quad (3)$$

Case 1 $F(p)$ simple. This means, at $p$,

$$F_{ab} = r_{asb} - s_{arb} \quad (r, s \text{ span blade of } F(p))$$

Case 2 $F(p)$ non-simple. This means, at $p$,

$$F_{ab} = \alpha(t_{azb} - z_{ab}) + \beta(x_0 y_b + y_a x_b) \quad (\alpha, \beta \in \mathbb{R})$$

Here there are two orthogonal canonical blades of $F(p)$ spanned by $t$ and $z$ and by $x$ and $y$.

Then it follows from (3) that if $F(p)$ is simple, each non-zero vector in the blade of $F(p)$ is an eigenvector of $R_{ab}$ with the same eigenvalue (i.e., the blade is an eigenspace of $R_{ab}$).

Similar remarks apply in the non-simple case to each blade (but with, in general, different eigenvalues).
5 Local Killing Vector Fields.

So far we have considered Killing vector fields defined \underline{globally} on $M$, although their local flows were \underline{local} isometries.

But physics suggests that an observer will only be able to observe Killing vector fields in his neighbourhood.

So suppose, instead of Killing vector fields defined \underline{globally} on $M$, we have, for each $p \in M$, an open neighbourhood $U$ of $p$ and a Lie algebra $K(U)$ of Killing vector fields defined on $U$. The question is: are these Killing vector fields on $U$ merely restrictions, to $U$, of \underline{global} Killing vector fields on $M$?

The answer depends on the choice of $U$, the dimensions of the various $K(U)$ and the \underline{topology} of $M$. 
6. **Curvature Symmetry.**

Suppose now that $X$ is such that its local flows are curvature tensor preserving. Then $X$ is called a **curvature collineation**, and

$$L_X R^a_{bcd} = 0.$$  

Denote the Lie algebra of smooth curvature collineations on $M$ by $\mathcal{CC}(M)$.

**Example**

$$ds^2 = -dt^2 + h_{\alpha\beta} dx^\alpha dx^\beta$$

$$[x, \beta, \gamma = 1, 2, 3, \ t = x^4, \ h_{\alpha\beta} = h_{\alpha\beta}(x^\gamma)]$$

Then $f(t) \frac{\partial}{\partial t}$ ($= (0, 0, 0, f(t))$ is in $\mathcal{CC}(M)$ for any smooth $f$.

**Problems!**

(i) Differentiability  
(ii) Dimension of $\mathcal{CC}(M)$  
(iii) Orbit Structure  
(iv) Vanishing on open subsets of $M$.

**Result** "In general" $\mathcal{CC}(M) = \mathcal{H}(M)$.
Suppose, in some coordinate system, we have metrics $g_{ab}$ and $\tilde{g}_{ab}$ with the property that their type (1,3) curvature tensors are equal:

$$R^a_{\;bcd} = \tilde{R}^a_{\;bcd}.$$ 

Then, generically, $g_{ab} = c \tilde{g}_{ab}$ where $c$ is constant.

So, generically, we have the following:

and where $\text{Riem}$ is the curvature tensor of $g$ and $\phi_t^* \text{Riem} = c \phi_t^* g$

So, if $X$ is a curvature collineation, $g$ and $\phi_t^* g$ have same $\text{Riem}$ and $\phi_t^* g = c g$ (c constant)