RECENT APPLICATIONS
OF THE
WEYL ANOMALY

ISLAMABAD
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33 YEARS OF THE WEYL ANOMALY

1973: 2 SALAM PROTEGES (CAPPERS & DUFF) ON FIRST POSTDOCS IN TRIESTE:

CONFORMAL INVARIANCE UNDER WEYL RESCALING OF THE METRIC TENSOR

\[ g^{\mu\nu}(x) \rightarrow \Omega^2(x) g^{\mu\nu}(x) \]

NO LONGER SURVIVE IN THE QUANTUM THEORY \( \Rightarrow \) DESER, DUFF & ISHAMI

\[ g^{\mu\nu}(T_{\mu\nu}) = cF - aG + 2\Omega R = \frac{\Delta}{2} \]

\[ F = C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \]

\[ G = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \]
CFT COUPLED TO GRAVITY

\[ a = \frac{1}{720 (4\pi)^2} \left[ 2N_0 + 11N_{1/2} + 124N_1 \right] \]

\[ c = \frac{3}{2} a = \frac{1}{120 (4\pi)^2} \left[ N_0 + 3N_{1/2} + 12N_1 \right] \]

\[ N_S = \text{NUMBER OF FIELDS WITH SPIN } S \]

\( a \) AND \( c \) ARE SCHEME-INDEPENDENT

BUT \( a \) IS NOT (e.g. \( 12N_1 \) DIM REQ \( -18N_1 \) ZETA FN)

CAN ALWAYS ADJUST BY ADDING \( \frac{\sqrt{g} \, R^2}{\pi} \)

INTERESTING SPECIAL CASE \( N=4 \) Y.M.

\[ U(n) : (N_1, N_{1/2}, N_0) = (N^2, 4N^2, 6N^2) \]

\[ a = c = N^2 / 4 (4\pi)^2 \]
\( N = 4 \) U(\( N \)) YANG-MILLS

INTERESTING SPECIAL CASE OF CFT

\((N_1, N_{1/2}, N_0) = (N^2, 4N^2, 6N^2)\)

\[ a = c = \frac{N^2}{64\pi^2} \]

\[ \mathcal{A} = \frac{N^2}{32\pi^2} \left( R^{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right) \]
TWENTY YEARS OF
THE WEYL ANOMALY

DEDICATED TO ABDUS SALAM

TRIESTE 93

BLACK HOLE PHYSICS
COSMOLOGY
STRING THEORY
STATISTICAL MECHANICS
RECENT APPLICATIONS

1998 THE "HOLONOMIC" WEYL ANOMALY IN ADS/CFT CORRESPONDENCE

2000 WEYL ANOMALY AND CORRECTIONS TO NEWTON'S LAW: COMPLEMENTARITY OF THE MALDAKEN AND RANDALL-SUNDRUNUM PICTURES

2000 REVIVAL OF (STAROBINSKY) WEYL ANOMALY DRIVEN INFLATION

2002 WEYL ANOMALY AND THE GRAVITON MASS IN ADS: COMPLEMENTARITY OF THE MALDAKEN AND KARCH-RANDALL PICTURES

2005 INFRA-RED DIAGNOSTIC
Holographic Weyl Anomaly

\[ \text{Gravity on (d+1) dim manifold } M \]
\[ \equiv \text{CFT on d dim manifold } \partial M \]

\[ Z_{\text{grav}} [\phi_0] = \int d\phi \exp(-S[\phi]) \]
\[ \phi_0 \]

\[ \phi \text{ on } M \Rightarrow \phi_0 \text{ on } \partial M \text{ correspond to } 0 \]

\[ Z_{\text{CFT}} [\phi_0] = \langle \exp \int d^dx \phi_0 \phi_0 \rangle_{\partial M} \]

\[ Z_{\text{grav}} = Z_{\text{CFT}} \]
\[
S = \frac{1}{16 \pi G} \int d^{d+1}x \sqrt{\det g_{\mu \nu}} \left( R + 2\Lambda \right)
\]

DIVERGES WHEN EVALUATED AT SOLUTIONS \( g_{\mu \nu} \) OF

\[
R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = \Lambda g_{\mu \nu}
\]

COORDINATE CHOICE

\[
g_{\mu \nu} dx^\mu dx^\nu = \frac{L^{2 + 1}}{4} g_{ij} dx^i dx^j + g_{ij} \theta_{ij} dx^i dx^j
\]

\[
g_{\mu \nu} g(x, y) = g(0) g(x) + g(1) g(x) + \ldots + g(2) g(x) + g(1) \ln g(x) + \ldots
\]

\[
\mathcal{O}[g_0] = \frac{1}{16 \pi G} \int d^4x \sqrt{g(0)} \left[ -\frac{d-12}{2} - \frac{d-12+1}{2} \right.
\]

\[
+ \ldots + \varepsilon a_2 a_0 + \ln \varepsilon a_2 + \text{finite}
\]
WEYL ANOMALY

DIVERGENCEs CANCELLED BY COUNTERTERMS. BREAKS $D=5$

GENERAL COVARIANCE $\Rightarrow$

$D=4$ ANOMALY

HENNINGSON & SKENDERIS

$\delta g(0) = 2 \delta \sigma g(0)$

$\delta S = \int d^4 x \sqrt{g_0} \chi \delta \sigma$

$\chi = \frac{1}{16 \pi G_{\text{eff}}} \left( -2 \alpha(\alpha) \right)$

EXAMPLE $D=4$

$\chi = \frac{L s^2}{64 \pi G_5} \left( R_{\mu \nu} R^{\mu \nu} - \frac{1}{3} R^2 \right)$
AdS$_5 \times S^5$ Type IIB

Geometry of $N$ coincident 3-Branes

\[ N^2 = \frac{\pi L_5^3}{2 \, G_5} \]

Maldacena: Dual to $N=4 \ U(n)$ Yang-Mills

\[ \mathcal{A} = \frac{N^2}{32 \pi^2} \left( R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right) \]

Same as 1-loop CFT anomaly (non-renormalization theorem)
I loop corrections to Newton's law and with Salam.

\[ V(r) = \frac{GM}{r} \left( 1 + \frac{\alpha G_4}{r^2} + \ldots \right) \]  

HJO 1972

\[ \alpha = \frac{1}{45\pi} \left( 12n_1 + 3n_{1/2} + n_0 \right) = \frac{8(4\pi)^2 c}{3\pi} \]

(same coefficient as in Weyl anomaly)

Maldacena N=4 CFT

\[ (n_1, n_{1/2}, n_0) = (N^2, 4N^2, 6N^2) \]

\[ \pi N^2 = \pi L s^3 / 2G_5 \quad G_4 = 2G_5 / Ls \]

\[ V(r) = \frac{G_4 M}{r} \left( 1 + \frac{2Ls^2}{3r^2} \right) \]

Same as Randall-Sundrum using D=5 classical calculation!

Same \( \alpha \) appears in graviton mass!
ANOMALY-DRIVEN INFLATION

FOR LARGE NUMBER OF MATTER FIELDS, CAN NEGLECT GRAVITON LOOPS

\[ R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = 8\pi G_4 \langle T_{\mu \nu} \rangle \]

MATTER EFFECTIVELY CFT IF \( m^2 \ll R \) BUT ANOMALY

\[ \lambda = g_{\mu \nu} \langle T_{\mu \nu} \rangle = 0 \]

MEANS FOR MAXIMALLY SYMMETRIC SPACES

\[ \langle T_{\mu \nu} \rangle = \frac{1}{4} g_{\mu \nu} \lambda \]

EFFECTIVE COSMOLOGICAL CONSTANT \( \Rightarrow (UNSTABLE) \) DE SITTER INFLATION

STAROBINSKY 1980
REVIVAL
VILENKIN 1985
HAWKING HERTOG & REALL 2000

1. Looked at $\mathcal{N}=4$ $V(n)$ Yang Mills in Large $N$ Limit $\Rightarrow$ AdS/CFT

2. Universe enters de Sitter phase through tunnelling from "nothing" via cosmological instanton

3. Get agreement with CMB data at expense of fine-tuning coefficient of $\sqrt{g} R^2$

Counterterm (without constraining $N$)
**Graviton Mass in AdS$_4$**

**Karch-Randall: AdS$_4$ Brane in AdS$_5$ Bulk ⇒**

Graviton Mass

\[ M^2 = \frac{3 L_5^2}{2 L_4^2} \]

$L_5 =$ radius of AdS$_5$

$L_4 =$ radius of AdS$_4$

**AdS/CFT Interpretation**

Extra 3 degrees of freedom come from massive spin 1 bound state

\[ M^2 = \frac{96\pi G}{L_4} c \]

$c =$ Weyl anomaly

\[ = \frac{3n^2 G}{2\pi L_4^4} \] agrees with K-R result!