The derivative of the topological susceptibility at zero momentum and an estimate of $\eta'$ mass in the chiral limit

J. Pasupathy\textsuperscript{a}, J. P. Singh\textsuperscript{b}, R. K. Singh\textsuperscript{a} and A. Upadhyay\textsuperscript{b}

\textsuperscript{a} Center for High Energy Physics, Indian Institute of Science, Bangalore-560 012, India
\textsuperscript{b} Physics Department, Faculty of Science, M.S. University of Baroda, Vadodara-390 002, India
Abstract
The anomaly-anomaly correlator is studied using QCD sum rules. Using the matrix elements of anomaly between vacuum and pseudoscalars $\pi$, $\eta$ and $\eta'$, the derivative of correlator $\chi'(0)$ is evaluated and found to be $\approx 1.82 \times 10^{-3}$ GeV$^2$. Assuming that $\chi'(0)$ has no significant dependence on quark masses, the mass of $\eta'$ in the chiral limit is found to be $\approx 723$ MeV. The same calculation also yields for the singlet pseudoscalar decay constant in the chiral limit a value of $\approx 178$ MeV.
The axial vector current in QCD has an anomaly

\[ \partial^\mu \bar{q} \gamma_\mu \gamma_5 q = 2i \, m_q \, \bar{q} \gamma_5 q - \frac{\alpha_s}{4\pi} \, G^{a}_{\mu\nu} \tilde{G}^{a\mu\nu}, \quad \text{where,} \quad \tilde{G}^{a\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} G^a_{\rho\sigma}. \quad (1) \]

The topological susceptibility \( \chi(q^2) \) defined by

\[ \chi(q^2) = i \int d^4x \, e^{iq \cdot x} \langle 0 | T \{ Q(x), \, Q(0) \} | 0 \rangle, \quad \text{with,} \quad Q(x) = \frac{\alpha_s}{8\pi} \, G^{a}_{\mu\nu} \tilde{G}^{a\mu\nu} \quad (2) \]

\[ \chi'(0) = \left. \frac{d\chi(q^2)}{dq^2} \right|_{q^2=0} \]
Using dispersion relation one can write

\[ \frac{\chi'(q^2)}{q^2} - \frac{\chi'(0)}{q^2} = \frac{1}{\pi} \int ds \ \mathcal{S}(\chi(s)) \left[ \frac{1}{s(s-q_2)^2} + \frac{1}{s^2(s-q_2)^2} \right] + \text{subtractions}. \]

Defining the Borel transform of a function \( f(q^2) \) by

\[ \hat{B} f(q^2) = -q^2, n \to \infty \left[ \frac{(-q^2)^{n+1}}{n!} \left( \frac{d}{dq^2} \right)^n f(q^2) \right]_{-q^2/n=M^2} \]

one gets from Eq.(4)

\[ \chi'(0) = \frac{1}{\pi} \int ds \ \mathcal{S}(\chi(s)) \left( 1 + \frac{s}{M^2} \right) e^{-s/M^2} - \hat{B} \left[ \frac{\chi'(q^2)}{q^2} \right]. \]  \hspace{1cm} (6)

\( \mathcal{S}(\chi(s)) \) receives contribution from all states \( |n\rangle \) such that \( \langle 0|Q|n\rangle \neq 0 \).
\[ \langle 0 | Q | \pi^0 \rangle = i \, f_\pi \, m_\pi^2 \, \left( \frac{m_d - m_u}{m_d + m_u} \right) \frac{1}{2\sqrt{2}}. \]  

(7)

The matrix elements, when \( |n\rangle \) is \( |\eta\rangle \) or \( |\eta'\rangle \), can be determined as follows. It is known that theoretical considerations based on chiral perturbation theory as well as phenomenological arguments one needs two mixing angles \( \theta_8 \) and \( \theta_0 \) to describe the coupling of the octet and singlet axial currents to \( \eta \) and \( \eta' \) [7, 8, 9]. Introduce the definition

\[ \langle 0 | J^a_{\mu_5} | P(p) \rangle = i \, f_\mu^a \, p_\mu; \ a = 0, 8; \ P = \eta, \eta', \]

(9)

\[ J^8_{\mu_5} = \frac{1}{\sqrt{6}} \left( \bar{u} \gamma_\mu \gamma_5 u + \bar{d} \gamma_\mu \gamma_5 d - 2 \bar{s} \gamma_\mu \gamma_5 s \right) \]

(10)

\[ J^0_{\mu_5} = \frac{1}{\sqrt{3}} \left( \bar{u} \gamma_\mu \gamma_5 u + \bar{d} \gamma_\mu \gamma_5 d + \bar{s} \gamma_\mu \gamma_5 s \right). \]

The \( |P(p)\rangle \) represents either \( \eta \) or \( \eta' \) with momentum \( p_\mu \).

\[
\begin{pmatrix}
  f_\eta^8 & f_\eta^0 \\
  f_{\eta'}^8 & f_{\eta'}^0 \\
\end{pmatrix}
= 
\begin{pmatrix}
  f_8 \cos \theta_8 & -f_0 \sin \theta_0 \\
  f_8 \sin \theta_8 & f_0 \cos \theta_0 \\
\end{pmatrix}
\]

(11)

Escribano and Frere find,
with \( f_8 = 1.28 f_\pi \) (\( f_\pi = 130.7 \text{MeV} \)),

the other three parameters to be

\[
\begin{align*}
\theta_8 &= (-22.2 \pm 1.8)^\circ, \\
\theta_0 &= (-8.7 \pm 2.1)^\circ, \\
f_0 &= (1.18 \pm 0.04) f_\pi.
\end{align*}
\]

The divergence of the axial currents are given by

\[
\begin{align*}
\partial^\mu J_{\mu 5}^8 &= \frac{i}{\sqrt{6}} \left( m_u \bar{u}\gamma_5 u + m_d \bar{d}\gamma_5 d - 2m_s \bar{s}\gamma_5 s \right) \\
\partial^\mu J_{\mu 5}^0 &= \frac{i}{\sqrt{3}} \left( m_u \bar{u}\gamma_5 u + m_d \bar{d}\gamma_5 d + m_s \bar{s}\gamma_5 s \right) + \frac{1}{\sqrt{3}} \frac{3\alpha_s}{4\pi} G_{\mu \nu}^a \tilde{G}^{a \mu \nu} 
\end{align*}
\]

(14) \hspace{1cm} (15)

Since \( m_u, m_d \ll m_s \), one can neglect them \cite{10} to obtain

\[
\begin{align*}
\langle 0 | \frac{3\alpha_s}{4\pi} G_{\mu \nu}^a \tilde{G}^{a \mu \nu} | \eta \rangle^{3/2} &= \frac{3}{2} m_\eta \left( f_8 \cos \theta_8 - \sqrt{2f_0} \sin \theta_0 \right) \\
\langle 0 | \frac{3\alpha_s}{4\pi} G_{\mu \nu}^{a \prime} \tilde{G}^{a \prime \mu \nu} | \eta' \rangle^{3/2} &= \frac{3}{2} m_{\eta'} \left( f_8 \sin \theta_8 + \sqrt{2f_0} \cos \theta_0 \right) .
\end{align*}
\]

(16) \hspace{1cm} (17)

Using Eqs.\((7)\), \((16)\) and \((17)\) we get the representation of \( \chi(q^2) \) in terms of physical states as

\[
\begin{align*}
\chi(q^2) &= -\frac{m_\pi^4}{8(q^2 - m_\pi^2)} f_\pi^2 \left( \frac{m_d - m_u}{m_d + m_u} \right)^2 - \frac{m_\eta^4}{24(q^2 - m_\eta^2)} \left( f_8 \cos \theta_8 - \sqrt{2f_0} \sin \theta_0 \right)^2 \\
&\quad - \frac{m_{\eta'}^4}{24(q^2 - m_{\eta'}^2)} \left( f_8 \sin \theta_8 + \sqrt{2f_0} \cos \theta_0 \right)^2 + \text{higher mass states.}
\end{align*}
\]

(18)
On the other hand, $\chi(q^2)$ has an operator product expansion [11, 12, 1, 5]

$$
\chi(q^2)_{OPE} = -\left(\frac{\alpha_s}{8\pi}\right)^2 \frac{2}{\pi^2} q^4 \ln \left(\frac{-q^2}{\mu^2}\right) \left[ 1 - \frac{\alpha_s}{\pi} \frac{83}{4} - \frac{9}{4} \ln \left(\frac{-q^2}{\mu^2}\right) \right]
$$

$$
- \frac{1}{16} \frac{\alpha_s}{\pi} \langle 0 | \frac{\alpha_s}{\pi} G^2 | 0 \rangle \left( 1 - \frac{9}{4} \frac{\alpha_s}{\pi} \ln \left(\frac{-q^2}{\mu^2}\right) \right) + \frac{1}{8q^2} \frac{\alpha_s}{\pi} \langle 0 | \frac{\alpha_s}{\pi} g_s G^3 | 0 \rangle
$$

$$
- \frac{15}{128} \frac{\pi \alpha_s}{q^4} \langle 0 | \frac{\alpha_s}{\pi} G^2 | 0 \rangle^2 + 16 \left(\frac{\alpha_s}{4\pi}\right)^3 \sum_{i=u,d,s} m_i \langle \bar{q}_i q_i \rangle \left[ \ln \left(\frac{-q^2}{\mu^2}\right) + \frac{1}{2} \right]
$$

$$
- \left[ \frac{q^4}{2} \int d\rho \, n(\rho) \rho^4 K_2^2(Q\rho) + \text{screening correction to the direct instantons} \right]. \tag{19}
$$
From Eq.(6), we now obtain

\[ \chi'(0) = \frac{f_\pi^2}{8} \left( \frac{m_d - m_u}{m_d + m_u} \right)^2 \left( 1 + \frac{m_\pi^2}{M^2} \right) e^{-\frac{m_\pi^2}{M^2}} + \frac{1}{24} \left( f_8 \cos \theta_8 - \sqrt{2} f_0 \sin \theta_0 \right)^2 \left( 1 + \frac{m_{\eta'}^2}{M^2} \right) e^{-\frac{m_{\eta'}^2}{M^2}} + \frac{1}{24} \left( f_8 \sin \theta_8 + \sqrt{2} f_0 \cos \theta_0 \right)^2 \left( 1 + \frac{m_{\eta'}^2}{M^2} \right) e^{-\frac{m_{\eta'}^2}{M^2}} \]

\[- \left( \frac{\alpha_s}{4\pi} \right)^2 \frac{1}{\pi^2} M^2 E_0(W^2/M^2) \left[ 1 + \frac{\alpha_s}{\pi} \frac{7}{4} + \frac{\alpha_s}{\pi} \frac{9}{2} \left( \gamma - \ln \frac{M^2}{\mu^2} \right) \right] \]

\[-16 \left( \frac{\alpha_s}{4\pi} \right)^3 \frac{1}{M^2} \sum_{i=u,d,s} m_i (\bar{q}_i q_i) - \frac{9}{64 M^2} \left( \frac{\alpha_s}{\pi} \right)^2 \left( \frac{\alpha_s}{\pi} \right)^2 \left( \frac{G^2}{\pi} \right) \]

\[+ \frac{1}{16} \frac{\alpha_s}{M^4} \left( \frac{G_3}{\pi} \right) - \frac{5}{128} \frac{\pi^2 \alpha_s}{M^6} \left( \frac{G^2}{\pi} \right)^2. \]
Here $E_0(x) = 1 - e^{-x}$ and takes into account the contribution of higher mass states, which has been summed using duality to the perturbative term in $\chi OPE$, and $W$ is the effective continuum threshold. We take $W^2 = 2.3$ GeV$^2$, and Fig.1 plot the r.h.s. of Eq.(20) as a function of $M^2$. We take $\alpha_s = 0.5$ for $\mu = 1$ GeV and

$$\langle 0|g_s^2 G^2|0\rangle = 0.5 \text{ GeV}^2, \quad \langle 0|\bar{s}s|0\rangle = 0.8\langle 0|\bar{u}u|0\rangle \quad \text{with} \quad \langle 0|\bar{u}u|0\rangle = -(240 \text{ MeV})^3,$$

$$m_s = 150 \text{ MeV} \quad \text{and} \quad m_u/m_d \approx 0.5. \quad (21)$$

Writing

$$\langle 0|g_s^3 G^3|0\rangle = \frac{\epsilon}{2} \langle 0|g_s^2 G^2|0\rangle, \quad (22)$$

as in Ref. [5], we take $\epsilon = 1$ GeV$^2$. We also have the PCAC relation,

$$-2(m_u + m_d) \langle 0|\bar{u}u|0\rangle = f_\pi^2 m_\pi^2. \quad (23)$$

For $f_0, f_8, \theta_8$ and $\theta_0$ we use the central values given in Eqs.(12) and (13).
Fig. 1: Various terms contributing to $\chi'(0)$, Eq.(20). The value of $\chi'(0)$ is the one obtained without the direct instantons. The latter, see Eq.(29), is given by $\chi_{DI}'$, which is larger than $\chi_{OPE}'$ and also has the wrong $M^2$ behaviour suggesting that screening corrections are important.
\[ \chi'(0) \approx 1.82 \times 10^{-3} \text{ GeV}^2. \]

We note that the above determination, Eq.(24), is in agreement with an entirely different calculation by two of us [14] from the study of the correlator of isoscalar axial vector currents

\[
\begin{align*}
\Pi^{I=0}_{\mu\nu} & = \frac{i}{2} \int d^4x \ e^{iq \cdot x} \langle 0 \mid \{ \bar{u} \gamma_\mu \gamma_5 u(x) + \bar{d} \gamma_\mu \gamma_5 d(x), \bar{u} \gamma_\mu \gamma_5 u(0) + \bar{d} \gamma_\mu \gamma_5 d(0) \} \mid 0 \rangle \\
\Pi^{I=0}_{\mu\nu} & = -\Pi^{I=0}_{1}(q^2) g_{\mu\nu} + \Pi^{I=0}_{2}(q^2) q_\mu q_\nu.
\end{align*}
\]

(25)

\[ \Pi^{I=0}_{1}(q^2 = 0) \] can be computed from the spectrum of axial vector mesons. In Ref. [14] a value

\[ \Pi^{I=0}_{1}(q^2 = 0) = -0.0152 \text{ GeV}^2 \]

It is not difficult to see that when \( m_u = m_d = 0 \)

\[ \chi'(0) = -\frac{1}{8} \Pi^{I=0}_{1}(q^2 = 0) \]
On the other hand, $\chi(q^2)$ has an operator product expansion [11, 12, 1, 5]

\[
\chi(q^2)_{OPE} = -\left(\frac{\alpha_s}{8\pi}\right)^2 \frac{2}{\pi^2} q^4 \ln\left(\frac{-q^2}{\mu^2}\right) \left[1 + \frac{\alpha_s}{\pi} \left(\frac{83}{4} - \frac{9}{4} \ln\left(\frac{-q^2}{\mu^2}\right)\right)\right] \\
- \frac{1}{16} \frac{\alpha_s}{\pi} \langle 0| \frac{\alpha_s}{\pi} G^2 |0\rangle \left(1 - \frac{9}{4} \frac{\alpha_s}{\pi} \ln\left(\frac{-q^2}{\mu^2}\right)\right) + \frac{1}{8q^2} \frac{\alpha_s}{\pi} \langle 0| \frac{\alpha_s}{\pi} g_s G^3 |0\rangle \\
- \frac{15}{128} \frac{\pi\alpha_s}{q^4} \langle 0| \frac{\alpha_s}{\pi} G^2 |0\rangle^2 + 16 \left(\frac{\alpha_s}{4\pi}\right)^3 \sum_{i=u,d,s} m_i \langle \bar{q}_i q_i\rangle \left[\ln\left(\frac{-q^2}{\mu^2}\right) + \frac{1}{2}\right] \\
- \left[\frac{q^4}{2} \int d\rho \, n(\rho) \, \rho^4 \, K_2^2(Q\rho) + \text{screening correction to the direct instantons}\right].
\]
\[ n(\rho) = n_0 \, \delta(\rho - \rho_c) \]

with \( n_0 = 0.75 \times 10^{-3} \text{ GeV}^4 \) and \( \rho_c = 1.5 \text{ GeV}^{-1} \). The contribution of the direct instanton to \( \hat{B}[\chi'(q^2)/q^2] \) can be found using the asymptotic expansion for \( K_2(z) \) and \( K'_2(z) \) and we find it to be

\[ \chi'_{DI} = \frac{n_0}{4} \sqrt{\pi} \, \rho_c^4 \, M^2 \left[ M \rho_c + \frac{9}{4} \frac{1}{M \rho_c} + \frac{45}{32} \frac{1}{M^3 \rho_c^3} \right] e^{-M^2 \rho_c^2}. \]
Fig. 1: Various terms contributing to $\chi'(0)$, Eq.(20). The value of $\chi'(0)$ is the one obtained without the direct instantons. The latter, see Eq.(29), is given by $\chi_{DI}$, which is larger than $\chi_{OPE}$ and also has the wrong $M^2$ behaviour suggesting that screening corrections are important.
We now turn to an estimate of $\eta'$ mass in the chiral limit: $m_u = m_d = m_s = 0$. 

$SU(3)$ flavor symmetry is exact and, we have $m_{\pi} = m_{\eta} = 0$ while $\eta'$ is a singlet.

Let us denote by

$$\eta_{\chi} = \eta'(m_s = 0) \text{ and } m_{\chi} = m_{\eta'}(m_s = 0),$$

we first note that the explicitly quark mass dependent term in $\chi O P E$

$$-16 \left( \frac{\alpha_s}{4\pi} \right)^3 \sum_{i=u,d,s} m_i \langle \bar{q}_i q_i \rangle \approx 1.9 \times 10^{-6} \text{ GeV}^4$$

is numerically much smaller than, for example

$$\frac{9}{64} \left( \frac{\alpha_s}{\pi} \right)^2 \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle \approx 4.5 \times 10^{-5} \text{ GeV}^4$$

which itself is much smaller than the perturbative term.
In the chiral limit $\langle 0|Q|\pi \rangle = \langle 0|Q|\eta \rangle = 0$. If we assume that the quark mass dependence of $\chi'(0)$ is negligible then $\chi'(0)$ in Eq.(20) can also be expressed in terms of $f_{\eta\pi}$ and $m_\pi$ as:

$$
\chi'(0) = \frac{1}{12} f_{\eta\pi}^2 \left( 1 + \frac{m_\chi^2}{M^2} \right) e^{-\frac{m_\chi^2}{M^2}} - \hat{B} \left[ \frac{\chi'_{OPE}(q^2)}{q^2} \right].
$$

$$
\frac{1}{12} f_{\eta\pi}^2 \left( 1 + \frac{m_\chi^2}{M^2} \right) e^{-\frac{m_\chi^2}{M^2}} \approx \frac{1}{24} f_\pi^2 \left( 1 + \frac{m_\pi^2}{M^2} \right) e^{-\frac{m_\pi^2}{M^2}}
+ \frac{1}{24} \left( f_8 \cos \theta_8 - \sqrt{2} f_0 \sin \theta_0 \right)^2 \left( 1 + \frac{m_\eta^2}{M^2} \right) e^{-\frac{m_\eta^2}{M^2}}
+ \frac{1}{24} \left( f_8 \sin \theta_8 + \sqrt{2} f_0 \cos \theta_0 \right)^2 \left( 1 + \frac{m_{\eta'}^2}{M^2} \right) e^{-\frac{m_{\eta'}^2}{M^2}}.
$$
We find $m_\chi \approx 723$ MeV and corresponding $f_{\eta\chi} = 178$ MeV.